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Incomplete perfect Mendelsohn designs with block size four

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Abstract

Let v, k, λ and n be positive integers. $(x_1,x_2,\ldots,x_k)$ is defined to be $\{(x_i,x_j)\colon i\neq j,\ i,j=1,2,\ldots,k\}$, in which the ordered pair (x_i,x_j) is called $(j-i)$-apart for $i<j$ and $(k-j+i)$-apart for $i>j$, and is called a cyclically ordered k-subset of $\{x_1,x_2,\ldots,x_k\}$. An incomplete perfect Mendelsohn design, denoted by (v,n,k,λ)-IPMD is a triple $(X,Y,\mathscr{B})$, where X is a v-set (of points), Y is an n-subset of X, and B is a collection of cyclically ordered k-subsets of X (called blocks), such that every ordered pair $(a,b)\in(X\times X)\backslash(Y\times Y)$ appears t-apart in exactly λ blocks of $\mathscr{B}$ and no pair $(a,b)\in Y\times Y$ appears in any block of $\mathscr{B}$ for any t, where $1\leqslant t\leqslant k-1$. In this article, we shall show that the necessary conditions $v(v-1)-n(n-1)\equiv 0\,(\mathrm{mod}\,4), v\geqslant 3n+1$ for the existence of a $(v,n,4,1)$-IPMD $(n\geqslant 2)$ are sufficient except for $n=2$, $v-n=5$ and possibly excepting $n=2$, $v-n=17,25$. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let v, u, k and λ be positive integers. $(x_1,x_2,\ldots,x_k)$ is defined to be $\{(x_i,x_j)\colon i\neq j,\ i,j=1,2,\ldots,k\}$, in which the ordered pair (x_i,x_j) is called $(j-i)$-apart for $i<j$ and $(k-j+i)$-apart for $i>j$, and is called a cyclically ordered k-subset of $\{x_1,x_2,\ldots,x_k\}$. An incomplete holey perfect Mendelsohn design, denoted by (v,u,k,λ)-IHPMD (k-IHPMD if $\lambda=1$) is a quadruple $(X,Y,\mathscr{G},\mathscr{A})$ which satisfies the following properties:

(i) X is a v-set (of points), Y is a u-subset of X;
(ii) $\mathscr{G}$ is a partition of X into groups;

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(iii) $\mathscr{A}$ is a collection of cyclically ordered k-subsets of X (called blocks) each of which intersects each group in at most one point;
(iv) No block contains two points of Y;
(v) Every ordered pair (x, y) from distinct groups such that at least one of x, y is in $X\backslash Y$ appears t-apart in exactly λ blocks of $\mathscr{A}$ for $t = 1, 2, \ldots, k-1$.

If $\mathscr{G} = \{G_i\colon 1 \leqslant i \leqslant h\}$, $|G_i| = g_i$, and $|G_i \cap Y| = u_i$, we say that $((g_1, u_1), (g_2, u_2), \ldots, (g_h, u_h))$ is the type of the IHPMD.

A (v, k, λ)-HPMD (k-HPMD if $\lambda = 1$) can be viewed as an IHPMD with $Y = \emptyset$ and the vector $(g_1, g_2, \ldots, g_h)$ is called the type of the HPMD.

A (v, n, k, λ)-IPMD can be viewed as an HPMD with the type of $(n, 1, 1, \ldots, 1)$ and a (v, k, λ)-PMD can be viewed as an IPMD with $n = 1$.

For more details on the above terminology, the reader is referred to [3,4,6,9].

A diamond incomplete perfect Mendelsohn design, denoted by $(v, w_1, w_2, w, k, \lambda)$-$\diamondsuit$-IPMD is a quadruple $(X, Y_1, Y_2, \mathscr{A})$, where X is a v-set, Y_1 is a w_1-subset of X, Y_2 is a w_2-subset of X, $w = |Y_1 \cap Y_2|$ and $\mathscr{A}$ is a collection of cyclically ordered k-subsets of X, such that every ordered pair (x, y) appears t-apart in exactly λ blocks of $\mathscr{A}$, unless $(x, y) \in Y_1$ or $(x, y) \in Y_2$, in which case the pair appears in no block, where $1 \leqslant t \leqslant k-1$ (see [9,11]). A (v, n, k, λ)-IPMD can be viewed as a $(v, n, 1, 0, k, \lambda)$-$\diamondsuit$-IPMD.

IPMDs are not only useful tools in the construction of PMDs (see [12]) and other structures, but finding when they exists is an interesting question itself (see [3–5, 9,13,14]).

The necessary conditions for the existence of a (v, k, λ)-IPMD were developed in [5], namely

$$\lambda(v-n)(v-(k-1)n-1) \equiv 0 \pmod{k}, \quad v \geqslant (k-1)n+1.$$

The basic necessary conditions were shown to be sufficient for the case $k = 3$ and $\lambda = 1$ with one exception of $v = 6$ and $n = 1$ (see [5]). For the case $k = 4$ and $\lambda > 1$, the basic necessary conditions were shown to be sufficient (see [9]). In the case of $k = 4$ and $\lambda = 1$, the basic necessary conditions are

$$v(v-1) - n(n-1) \equiv 0 \pmod{4}, \quad v \geqslant 3n+1,$$

which are equivalent to

$$v-n \equiv 0 \pmod{4}, \quad n = 0, 1, 2, \ldots, (v-n-2)/2,$$

$$v-n \equiv 1 \pmod{4}, \quad n = 0, 2, 4, \ldots, (v-n-1)/2,$$

$$v-n \equiv 3 \pmod{4}, \quad n = 1, 3, 5, \ldots, (v-n-1)/2.$$

If a pair $(v-n, n)$ satisfies the basic necessary conditions, then we say that the pair $(v-n, n)$ is admissible.

The existence of $(v, 4, 1)$-PMD forms the basis for most of our constructions. The problem of existence was initially studied by Mendelsohn (see [10]), and now we have a complete result in the form of the following theorem (see [1,2,4,10,12]).

Theorem 1.1. *A* $(v,4,1)$*-PMD exists if and only if* $v(v-1)\equiv 0\,(\mathrm{mod}\,4)$ *with the exception of* $v=4$ *and* 8.

The following results are from [3,4,6,13].

Theorem 1.2 (Bennett et al. [4]). *The necessary conditions* $v\equiv 2,3\,(\mathrm{mod}\,4)$ *and* $v\geqslant 7$ *for the existence of a* $(v,2,4,1)$*-IPMD are sufficient except for* $v=7$ *and possibly excepting* $v-n=12,13,16,17,20,21,24,25,28$.

Theorem 1.3 (Bennett et al. [3,4]). *The necessary conditions* $v\equiv 2,3\,(\mathrm{mod}\,4)$ *and* $v\geqslant 10$ *for the existence of a* $(v,3,4,1)$*-IPMD are sufficient*; *The necessary conditions* $v\equiv 2,3\,(\mathrm{mod}\,4)$ *and* $v\geqslant 22$ *for the existence of a* $(v,7,4,1)$*-IPMD are sufficient.*

Theorem 1.4 (Xuebin Zhang [13]). *There exists a* $(v,n,4,1)$*-IPMD for* $v-n=16,20,24,28$, $n=2$ *and* $v-n=12,16,20,\ldots,32$ *and* $4\leqslant n\leqslant (v-n-2)/2$.

Theorem 1.5 (Bennett and Xuebin Zhang [6]). *The necessary condition for the existence of a* $(v,4,1)$*-HPMD of type* m^h, *namely*, $h(h-1)m^2\equiv 0\,(\mathrm{mod}\,4)$ *is also sufficient with the exception of types* 2^4 *and* 1^8, *and type* m^4 *for odd* m.

A $(v,n,4,1)$-IPMD can be viewed as a $(v,n,1,0,4,1)$-$\Diamond$-IPMD, so we have

Lemma 1.6. *If there exists a* $(3m+1,m,4,1)$*-IPMD then there exists a* $(3m+1,m,1,0,4,1)$*-*$\Diamond$*-IPMD.*

In this article, we shall show that the necessary conditions $v(v-1)-n(n-1)\equiv 0\,(\mathrm{mod}\,4)$, $v\geqslant 3n+1$ for the existence of a $(v,n,4,1)$-IPMD $(n\geqslant 2)$ are sufficient except for $n=2$, $v-n=5$ and possibly excepting $n=2$, $v-n=17,25$.

We assume that the reader is familiar with the basic concepts in design theory, such as pairwise balanced design (PBD), group divisible design (GDD), incomplete group divisible design (IGDD), transversal design (TD), and incomplete transversal design (ITD). For convenience, the reader can be referred to [7,8].

2. Direct construction methods

A direct construction using groups below is a variation of the method using difference sets in the construction of BIBDs. Instead of listing all of the blocks of a design, it suffices to give the group acting on a set of base blocks.

Let G be a group, H a subgroup of G. Let $B=(b_1,b_2,b_3,b_4)$ be a block of G, and $B'=(\infty,b_2,b_3,b_4)$ a block of $G\cup\{\infty\}$. By developing the base block B or B' under H, we can obtain a set of blocks, that is

$$\mathrm{dev}\,B=\{gB=(gb_1,gb_2,gb_3,gb_4)\colon\ g\in H\}$$

or

$$\mathrm{dev}\,B'=\{gB'=(\infty,gb_2,gb_3,gb_4)\colon\ g\in H\}.$$

When $H=G$, we define their t-apart difference sets $B(t)$, $B'(t)$ for $t=1,2$ as follows:

$$B(1)=(b_1^{-1}b_2, b_2^{-1}b_3, b_3^{-1}b_4, b_4^{-1}b_1),$$

$$B(2)=(b_1^{-1}b_3, b_2^{-1}b_4, b_3^{-1}b_1, b_4^{-1}b_2),$$

$$B'(1)=(b_2^{-1}b_3, b_3^{-1}b_4),$$

$$B'(2)=(b_2^{-1}b_4, b_4^{-1}b_2).$$

Lemma 2.1. *There exists a* $(v,n,4,1)$*-IPMD for* $v-n=21,\ n=2$.

Proof. Let G be the non-abelian group of order 21 in which $a^7=b^3=1$ and $a^2b=ba$. Develop the following six base blocks under G:

$$(\infty_1, 1, ba^3, a^5),\quad (\infty_2, 1, b^2a^3, a^3),\quad (1, a^6, a, ba^3),$$

$$(1, a, ba, ba^6),\quad (1, a^3, ba^6, b^2a^3),\quad (1, a^4, b^2a^6, ba^5).$$

It is readily checked that the union of the t-apart difference sets of the six base blocks is $G-\{1\}$ for $t=1,2$. □

Lemma 2.2. *There exist* $(v,u,4,1)$*-IHPMDs of the types of* $(3,1)^4, (3,1)^4(4,1)^1$ *and* $(3,1)^4(3,0)^1$.

Proof. For type $(3,1)^4$, take $Y=\{\infty_i\colon 0\leqslant i\leqslant 3\}$, $G_i=\{\infty_i, i, i+4\}$ for $0\leqslant i\leqslant 3$, $G=Z_8$, $H=\{0,4\}$ and develop the following base blocks under H:

$$(\infty_0,1,7,2),\quad (\infty_0,3,2,5),\quad (\infty_0,2,1,3),$$

$$(\infty_1,2,3,0),\quad (\infty_1,3,4,6),\quad (\infty_1,0,6,3),$$

$$(\infty_2,1,4,3),\quad (\infty_2,0,3,1),\quad (\infty_2,3,5,4),$$

$$(\infty_3,1,6,4),\quad (\infty_3,2,4,5),\quad (\infty_3,0,5,6).$$

For type $(3,1)^4(4,1)^1$, take $Y=\{\infty,0,3,6,9\}, G_i=\{i,4+i,8+i\}$ for $1\leqslant i\leqslant 4$, $G_5=\{\infty,\infty_1,\infty_2,\infty_3\}, G=Z_{12},\ H=\{0,2,\ldots,10\},\ M=\{0,3,6,9\}$ and develop the following base blocks under G:

$$(\infty_2,0,11,1),\quad (\infty_3,0,10,5)$$

and develop the following base blocks under H:

$$(\infty_1,0,5,10),\quad (\infty_1,1,2,3)$$

and develop the following base blocks under M:

$$(\infty,1,10,4),\quad (\infty,2,11,5),$$

and union the following blocks:

$$(1,4,7,10),\quad (2,5,8,11).$$

For type $(3,1)^4(3,0)^1$, take $Y=\{0,3,6,9\}$, $G_i=\{3i,1+3i,2+3i\}$ for $0\leqslant i\leqslant 3$, $G_5=\{\infty_1,\infty_2,\infty_3\}$, $G=Z_{12}$, $H=\{0,3,6,9\}$ and develop the following base blocks under H:

$$(\infty_1,6,1,5),\quad (\infty_1,1,6,4),\quad (\infty_1,11,5,6),$$

$$(\infty_2,6,2,10),\quad (\infty_2,10,6,5),\quad (\infty_2,2,4,6),$$

$$(\infty_3,6,10,4),\quad (\infty_3,2,6,11),\quad (\infty_3,1,11,6)$$

and union the following blocks:

$$(1,4,7,10),\quad (1,8,11,4),\quad (1,10,5,8),$$

$$(2,5,10,7),\quad (2,7,4,11),\quad (2,11,8,5).\qquad \square$$

From now, we always let $G=Z_m$, a cyclic addition group of order m. Let $B=(b_1,b_2,b_3,b_4)$ is a base block under G, then its 1-apart difference set is

$$B(1)=(b_2-b_1,b_3-b_2,b_4-b_3,b_1-b_4)$$

and its 2-apart difference set is

$$B(2)=(b_3-b_1,b_4-b_2,b_1-b_3,b_2-b_4).$$

Let $B=(\infty,b_2,b_3,b_4)$ is a base block of under G, then

$$B(1)=(b_3-b_2,b_4-b_3),$$

$$B(2)=(b_4-b_2,b_2-b_4).$$

For even m, we always let $H=\{2g\colon g\in G\}$ be a subgroup of G. If $A=(b_1,b_2,b_3,b_4)$ of a base block under H, then its 1-apart difference set is denoted by

$$A(1)=((b_2-b_1)^{c_1},(b_3-b_2)^{c_2},(b_4-b_3)^{c_3},(b_1-b_4)^{c_4})$$

and its 2-apart difference set is denoted by

$$A(2)=((b_3-b_1)^{c_1},(b_4-b_2)^{c_2},(b_1-b_3)^{c_3},(b_2-b_4)^{c_4}).$$

If $A=(\infty,b_2,b_3,b_4)$ of a base block under H, then

$$A(1)=((b_3-b_2)^{c_2},(b_4-b_3)^{c_3}),$$

$$A(2)=((b_4-b_2)^{c_2},(b_2-b_4)^{c_4}),$$

where $c_i=0$ if b_i is even and $c_i=1$ if b_i is odd.

For convenience, in the following, we present $B(1)$ and $A(1)$ instead of the base block B under G and A under H, (see Examples 1.7, 1.11, Lemma 2.2 in [13]).

Example 2.1. *There exists a* $(v,n,4,1)$*-IPMD for* $v-n=27$, $n=9$.

Proof. Let $X = G = Z_{27}$, and $Y = \{\infty_1, \infty_2, \ldots, \infty_9\}$. We present $\mathscr{B}(1)$ as follows:

$$(1,4,-1,-4),(2,5,7,13),(-13,11),(12,-8),(-12,-9),$$
$$(10,9),(-10,-7),(3,8),(-2,-11),(-3,-6),(-5,6).$$

Then we have $\mathscr{B}(2)$ as follows:

$$(5,3,-5,-3,),(7,12,-7,-12),(2,-2),(4,-4),(6,-6),(8,-8),$$
$$(10,-10),(11,-11),(13,-13),(9,-9),(1,-1).$$

Then we have $\mathscr{B}$ as follows:

$$(0,1,5,4),(0,2,7,-13),(\infty_1,0,-13,-2),(\infty_2,0,12,4),$$
$$(\infty_3,0,-12,6),(\infty_4,0,10,-8),(\infty_5,0,-10,10),(\infty_6,0,3,11),$$
$$(\infty_7,0,-2,-13),(\infty_8,0,-3,-9),(\infty_9,0,-5,1).$$

It is readily checked that $\bigcup\mathscr{B}(1) = \bigcup\mathscr{B}(2) = G - \{0\}$, and then dev$\,\mathscr{B}$ forms the blocks of a $(v,n,4,1)$-IPMD for $v-n=27$, $n=9$. □

Example 2.2. *There exists a* $(v,4,1)$*-HPMD of type* $11^4 6^1$.

Proof. Let $v-n=44$, $G_i = \{0+i, 4+i, \ldots, 40+i,\}$, $i=0,1,2,3$, $G_5 = \{\infty_1,\infty_2,\ldots,\infty_6\}$, $\mathscr{E} = \{(0,11,22,-11)+i\colon i=0,1,\ldots,10\}$.

We take the group $G = Z_{44}$, $H = \{2g\colon g \in G\}$, a subgroup of G, and present $\mathscr{B}(1)$, and $\mathscr{A}(1)$ as follows:

$$(1,2,-1,-2),(9,18,-9,-18),(6,19,-6,-19),$$
$$(7,14,-7,-14),(5,10,-5,-10),(22,-11),$$
$$(3^0,3^1),(-3^1,-3^0);(17^0,17^1),(-17^1,-17^0);(21^0,21^1),(-21^1,-21^0),$$
$$(13^0,13^1),(-13^1,-13^0),(15^0,15^1),(-15^1,-15^0).$$

Then we have $\mathscr{B}(2)$, and $\mathscr{A}(2)$ as follows:

$$(3,1,-3,-1),(-17,9,17,-9),(-19,13,19,-13),$$
$$(21,7,-21,-7),(15,5,-15,-5),(11,-11),$$
$$(6^0,-6^0),(-6^1,6^1);(-10^0,10^0),(10^1,-10^1);(-2^0,2^0),(2^1,-2^1);$$
$$(-18^0,18^0),(18^1,-18^1);(-14^0,14^0),(14^1,-14^1).$$

Then we have $\mathscr{B}$ as follows:

$$(0,1,3,2),(0,9,-17,18),(0,6,-19,19),$$
$$(0,7,21,14),(0,5,15,10),(\infty_1,0,22,11)$$

and $\mathscr{A}$ as follows:

$$(\infty_2, 0, 3, 6), (\infty_2, 1, -2, -5); (\infty_3, 0, 17, -10), (\infty_3, 1, -16, 11);$$

$$(\infty_4, 0, 21, -2), (\infty_4, 1, -20, 3);\ (\infty_5, 0, 13, -18),\ (\infty_5, 1, -12, 19);$$

$$(\infty_6, 0, 15, -14),\ (\infty_6, 1, -14, 15).$$

It is readily checked that $(\operatorname{dev}\mathscr{B}) \cup (\operatorname{dev}\mathscr{A}) \cup \mathscr{E}$ forms the blocks of a $(v, 4, 1)$-HPMD of type $11^4 6^1$. □

Let B be a base block under $G = Z_{v-n}$, where $v - n$ is even, and $B(1) = (a, b, c, d)$. If a and c are even and b and d are odd, or a and c are odd and b and d are even, or a, b, c and d are odd, then B is called *a special base block*, we indicate it with *. It follows from Lemmas 2.8 and 2.9 in [13] that we can replace a special base block with four blocks with ∞ under H such that total 1-apart difference set and total 2-apart difference set are not changed.

Example 2.3. A special base block B under $G = Z_{40}$ with its $B(1) = (1, 2, -1, -2)^*$. We have

$$B(2) = (3, 1, -3, -1), \quad B = (0, 1, 3, 2)$$

and we can replace the special base block B with the four following base blocks under H:

$$(\infty_1, 1, 2, 4), \quad (\infty_1, 0, 1, -1); \quad (\infty_2, 1, 0, -2), \quad (\infty_2, 0, -1, 1)$$

and their 1-apart difference sets:

$$(1^1, 2^0), \quad (1^0, -2^1); \quad (-1^1, -2^0), \quad (-1^0, 2^1)$$

equal to $B(1)$, and their 2-apart difference sets:

$$(3^1, -3^0), \quad (-1^0, 1^1); \quad (-3^1, 3^0), \quad (1^0, -1^1)$$

equal to $B(2)$.

Lemma 2.3. *There exists a* $(4k + a, k + a, k + a, a, 4, 1)$-$\Diamond$-*IPMD for*

(i) $k = 3, a = 0$; (ii) $k = 5, a = 0, 2$; (iii) $k = 7, a = 1, 3$;
(iv) $k = 9, a = 0, 2, 4$; (v) $k = 11, a = 1, 3, 5$.

Proof. Let $G = Z_{2k}$, $H = \{0, 2, \ldots, 2k-2\}$, a subgroup of G. Let $Y_1 = \{x_0, x_2, \ldots, x_{k-1}, \infty_k, \infty_{k+1}, \ldots, \infty_{k+a-1}\}$, $Y_2 = \{\infty_0, \infty_1, \ldots, \infty_{k+a-1}\}$. We present the base blocks $\mathscr{A}$ under H and $\mathscr{C}(1)$ under G or H for each case, where $x_0 + 2i = x_i$, $i = 0, 1, \ldots, k-1$.

(i) $k = 3, a = 0$.

$$\begin{aligned}\mathscr{A} = \{&(\infty_0, x_0, 1, 2), (\infty_0, 4, x_0, 5), (\infty_0, 3, 0, x_0); (\infty_1, x_0, 4, 1), (\infty_1, 2, x_0, 0),\\ &(\infty_1, 5, 3, x_0); (\infty_2, x_0, 2, 4), (\infty_2, 1, x_0, 3), (\infty_2, 0, 5, x_0)\},\end{aligned}$$

$\mathscr{C}(1) = \{(1^0, 2^1, -1^1, -2^0)\}$, that is, $\mathscr{C} = \{(0, 1, 3, 2)\}$ under H.

(ii) $k=5, a=0,2$.

$$\mathscr{A} = \{(\infty_0,x_0,2,1),(\infty_0,1,x_0,6),(\infty_0,0,3,x_0);(\infty_1,x_0,9,8),(\infty_1,5,x_0,7),(\infty_1,6,8,x_0);(\infty_2,x_0,8,5),(\infty_2,2,x_0,4),(\infty_2,7,9,x_0);(\infty_3,x_0,5,9),(\infty_3,4,x_0,0),(\infty_3,3,0,x_0);(\infty_4,x_0,1,4),(\infty_4,7,x_0,3),(\infty_4,2,6,x_0)\},$$

$\mathscr{C}(1)=(1,-2,5,-4)^*$, that is, for $a=0$,

$\mathscr{C}=\{(0,1,-1,4)\}$ under G; and for $a=2$,

$\mathscr{C}=\{(\infty_5,0,1,-1),(\infty_5,1,6,4),(\infty_6,0,-4,1),(\infty_6,1,-3,-2)\}$ under H.

(iii) $k=7, a=1,3$.

$$\mathscr{A} = \{(\infty_0,x_0,4,2),(\infty_0,9,x_0,11),(\infty_0,12,11,x_0);(\infty_1,x_0,1,0),(\infty_1,2,x_0,7),(\infty_1,3,6,x_0);(\infty_2,x_0,6,9),(\infty_2,12,x_0,10),(\infty_2,5,3,x_0);(\infty_3,x_0,2,6),(\infty_3,4,x_0,13),(\infty_3,1,5,x_0);(\infty_4,x_0,8,13),(\infty_4,8,x_0,12),(\infty_4,11,7,x_0);(\infty_5,x_0,5,10),(\infty_5,13,x_0,9),(\infty_5,4,0,x_0);(\infty_6,x_0,0,8),(\infty_6,10,x_0,3),(\infty_6,7,1,x_0)\},$$

$\mathscr{C}(1)=\{(2,-3,6,-5)^*,(1,7)\}$

(iv) $k=9,\ a=0,2,4$.

$$\mathscr{A} = \{(\infty_0,x_0,1,8),(\infty_0,12,x_0,3),(\infty_0,9,0,x_0);(\infty_1,x_0,12,16),(\infty_1,3,x_0,15),(\infty_1,12,1,x_0);(\infty_2,x_0,2,11),(\infty_2,2,x_0,14),(\infty_2,1,5,x_0);(\infty_3,x_0,10,6),(\infty_3,15,x_0,7),(\infty_3,2,13,x_0);(\infty_4,x_0,11,7),(\infty_4,10,x_0,0),(\infty_4,13,6,x_0);(\infty_5,x_0,4,10),(\infty_5,16,x_0,17),(\infty_5,3,9,x_0);(\infty_6,x_0,6,0),(\infty_6,8,x_0,13),(\infty_6,5,17,x_0);(\infty_7,x_0,9,17),(\infty_7,7,x_0,8),(\infty_7,14,4,x_0);(\infty_8,x_0,5,15),(\infty_8,11,x_0,16),(\infty_8,4,14,x_0)\},$$

$\mathscr{C}(1)=\{(1,3,-1,-3)^*,(2,5,-2,-5)^*\}$.

(v) $k=11, a=1,3,5$.

$$\mathscr{A} = \{(\infty_0,x_0,7,0),(\infty_0,2,x_0,9),(\infty_0,17,0,x_0);(\infty_1,x_0,8,1),(\infty_1,5,x_0,12),(\infty_1,8,19,x_0);(\infty_2,x_0,20,3),(\infty_2,11,x_0,0),(\infty_2,14,9,x_0);(\infty_3,x_0,1,12),(\infty_3,3,x_0,11),(\infty_3,16,18,x_0);(\infty_4,x_0,13,15),(\infty_4,10,x_0,18),(\infty_4,9,4,x_0);(\infty_5,x_0,4,11),(\infty_5,20,x_0,10),(\infty_5,13,17,x_0);(\infty_6,x_0,21,6),(\infty_6,7,x_0,19),(\infty_6,4,8,x_0);(\infty_7,x_0,14,20),(\infty_7,12,x_0,17),(\infty_7,7,13,x_0);(\infty_8,x_0,16,10),(\infty_8,6,x_0,15),(\infty_8,5,21,x_0);(\infty_9,x_0,3,21),(\infty_9,1,x_0,6),(\infty_9,18,14,x_0);(\infty_{10},x_0,5,19),(\infty_{10},15,x_0,2),(\infty_{10},2,16,x_0)\},$$

$\mathscr{C}(1)=\{(8,-2),(1,3,-1,-3)^*,(9,10,-9,-10)^*\}$. □

Let B_1 and B_2 be two base blocks under $G=Z_{v-n}$, where $v-n$ is odd, and

$$B_1(1)=(a,b,c,d),\quad B_2(1)=(e,f).$$

If $c=2b-a, d=-3b, e=a-4b, f=6b-a$, then we indicate $B_1(1)$ and $B_2(1)$ with $(a,b,c,d;e,f)^*$. It is easy to see that we can replace B_1 and B_2 with the following three base blocks under G:

$$(\infty_1, 0, a, a+b), \quad (\infty_2, 0, c, c+e), \quad (\infty_3, 0, d, d+f),$$

such that total 1-apart difference set and total 2-apart difference set are not changed.

Example 2.4. Two base blocks B_1 and B_2 with

$$B_1(1) = (-4, 1, 6, -3), \quad B_2(1) = (-8, 10)$$

for $v - n = 31$, that is

$$B_1 = (0, -4, -3, 3), \quad B_2 = (\infty, 0, -8, 2).$$

We can replace B_1 and B_2 with the following three base blocks under $G = Z_{31}$:

$$(\infty_1, 0, -4, -3), \quad (\infty_2, 0, 6, -2), \quad (\infty_3, 0, -3, 7),$$

such that total 1-apart difference set and total 2-apart difference set are not changed.

Let B_1 and B_2 be two base blocks under $G = Z_{v-n}$, where $v - n$ is odd, and

$$B_1(1) = (a, b, -a, -b), \quad B_2(1) = (c, d).$$

If $c = b - a, d = 2a - b$, then we indicate $B_1(1)$ and $B_2(1)$ with $((B_1(1); B_2(1))^*$.

It is easy to see that we can replace B_1 and B_2 with the following three base blocks under G:

$$(\infty_1, 0, a, a+b), \quad (\infty_2, 0, d, d-a), \quad (\infty_3, 0, c, c-b)$$

such that total 1-apart difference set and total 2-apart difference set are not changed.

Example 2.5. Two base block B_1 and B_2 with

$$B_1(1) = (1, -4, -1, 4), \quad B_2(1) = (-5, 6)$$

for $v - n = 27$, that is

$$B_1 = (0, 1, -3, -4), \quad B_2 = (\infty, 0, -5, 1).$$

We can replace B_1 and B_2 with the following three base blocks under $G = Z_{31}$:

$$(\infty_1, 0, 1, -3), \quad (\infty_2, 0, 6, 5), \quad (\infty_3, 0, -5, -1)$$

such that total 1-apart difference set and total 2-apart difference set are not changed.

By the above easy and effective ways, we can directly construct a lot of examples of IPMDs and HPMDs.

Theorem 2.4. *There exists a $(v, n, 4, 1)$-IPMD for admissible $(v - n, n) \in \{(23, 5),$ $(27, 5),$ $(27, 9-11), (31, 9-11), (17, 4-6), (21, 4-10), (25, 4-12), (29, 4-14)\}$.*

Proof. Let $X = G = Z_{v-n}$, and $Y = \{\infty_1, \infty_2, \ldots, \infty_n\}$. We present $\mathscr{B}(1)$ as follows. It is readily checked that dev $\mathscr{B}$ forms the blocks of the IPMD.

$v-n=23,\ n=5$:
$(1,3,10,9),(-2,-5,-7,-9),(11,4,2,6),(-11,-10),(8,-3),(5,-6),(7,-4),$
$(-1,-8)$;

$v-n=27,\ n=5$:
$(7,-8,5,-4),(1,3,10,13),(-2,-5,-13,-7),(-12,4,6,2),(-11,-1),(9,12),$
$(-6,-10),(11,-9),(8,-3)$.

$v-n=27,\ n=9,11$:
$(1,-4,-1,4;-5,6)^*,(2,5,7,13),(-13,11),(12,-8),(-12,-9),(10,9),(-10,-7),$
$(3,8),(-2,-11),(-3,-6)$.

$v-n=31,\ n=9,11$:
$(1,-4,-1,4;-5,6)^*,(5,2,7,-14),(-7,-3,-8,-13),(-15,13),(15,-11),(11,14),$
$(10,-2),(-12,-6),(8,9),(12,3),(-9,-10)$.

$v-n=17,\ n=4,6$:
$(-1,4,1,-4;5,-6)^*,(-7,3,6,-2),(7,-5),(-3,-8),(2,8)$.

$v-n=21,\ n=4,6$:
$(1,-4,-1,4;-5,6)^*,(7,-3,-9,5),(2,8,-2,-8),(10,9),(-6,-7),(-10,3)$.

$v-n=21,\ n=8,10$:
$(1,-4,-1,4;-5,6)^*,(9,-7),(3,5),(-3,-8),(-9,2),(8,7),(10,-6),(-2,-10)$.

$v-n=25,\ n=4$:
$(5,-3,7,-9),(-5,11,-12,6),(10,1,2,12),(-2,-7,-10,-6),(-8,3),(-11,-4),$
$(-1,8),(4,9)$.

$v-n=25,\ n=6,8$:
$(1,-4,-1,4;-5,6)^*,(5,-3,-9,7),(-2,-7,-10,-6),(-8,-11),(10,8),(2,-12),$
$(3,11),(12,9)$.

$v-n=25,\ n=10,12$:
$(1,-4,-1,4;-5,6)^*,(8,-2),(-10,-8),(-11,-6),(12,-3),(-12,2),(5,9),(-9,7),$
$(11,-7),(3,10)$.

$v-n=29,\ n=4,6,8$:
$(-4,1,6,-3;-8,10)^*,(-10,2,14,-6;11,-7)^*,(12,-2,-13,3),(-1,-5,-12,-11),$
$(4,5,13,7),(9,-14),(-9,8)$.

$v-n=29,\ n=10,12,14$:
$(-4,1,6,-3;-8,10)^*,(-10,2,14,-6;11,-7)^*,(13,-12),(3,-9),(12,8),(-14,4),$
$(-1,-11),(5,9),(7,-2),(-13,-5)$. □

The following lemma is due to Hantao Zhang.

Lemma 2.5. *There exists a* $(v,n,4,1)$*-IPMD for* $v-n=13,\ n=2$.

Proof. The blocks are

$(\infty_1,6,0,9),(\infty_1,9,1,2),(\infty_1,8,2,1),(\infty_1,4,3,10),(\infty_1,10,4,8),$
$(\infty_1,1,5,7),(\infty_1,7,6,5),(\infty_1,11,7,12),(\infty_1,12,8,3),(\infty_1,5,9,11),$
$(\infty_1,2,10,0),(\infty_1,3,11,6),(\infty_1,0,12,4),(\infty_2,2,0,1),(\infty_2,3,1,0),$
$(\infty_2,4,2,3),(\infty_2,5,3,2),(\infty_2,6,4,5),(\infty_2,7,5,4),(\infty_2,8,6,7),(\infty_2,9,7,8),$
$(\infty_2,11,8,9),(\infty_2,12,9,10),(\infty_2,1,10,11),(\infty_2,0,11,12),(\infty_2,10,12,6),$
$(0,6,1,12),(0,10,5,11),(0,5,6,8),(0,4,7,9),(0,2,8,7),(0,3,9,5),$
$(0,7,10,3),(0,8,11,4),(1,7,3,12),(1,8,4,11),(1,3,5,8),(1,9,6,10),$
$(1,6,9,4),(1,11,10,7),(1,4,12,5),(2,11,3,6),(2,7,4,10),(2,4,6,11),$
$(2,9,8,12),(2,5,10,9),(2,12,11,5),(2,6,12,7),(3,4,9,12),(3,8,10,6),$
$(3,7,11,9),(5,12,10,8).$ □

Theorem 2.6. *There exists a* $(v,n,4,1)$*-IPMD for admissible* $(v-n,n)$ *with* $v-n \leqslant 32$ *and* $n \geqslant 2$, *and possibly except* $(v-n,n) \in \{(17,2),(25,2)\}$.

Proof. From Theorem 1.3 and the following Table 1.

3. Recursive construction methods

When we start with a $(h,k,1)$-PMD and replaces each block with a TD(k,m) we can obtain a $(hm,k,1)$-HPMD of type m^h, this is the idea of Theorem 2.2 in [12]; similarly, when we start with a TD(k,m) and replace each block with a $(h,k,1)$-PMD, we also can obtain a $(hm,k,1)$-HPMD of type m^h, this is the idea of Theorem 2.4 in [12]. Based on these ideas and Wilson's fundamental constructions for GDD and IGDD (see Theorems 2.5 and 2.57 in [7]), we have the following theorems.

Let s and t be two functions from X to non-negative integers with $t(x) \leqslant s(x)$. If

$$A=\{x_1,x_2,\ldots,x_r\} \subseteq X,$$

then we denote

$$s_A=s(x_1)+s(x_2)+\cdots+s(x_r), \quad t_A=t(x_1)+t(x_2)+\cdots+t(x_r).$$

Theorem 3.1. *Suppose* $(X,\mathcal{G},\mathcal{A})$ *is a GDD with* $\lambda=1$ *and for every block* $A\in\mathcal{A}$, *we have a* $(s_A,k,1)$*-HPMD of type* $\{s(x)\colon x\in A\}$. *Then there exists a* $(s_X,k,1)$*-HPMD of type* $\{s_G\colon G\in\mathcal{G}\}$.

Theorem 3.2. *Suppose* $(X,\mathcal{G},\mathcal{A})$ *is a* $(v,k,1)$*-HPMD, and for every block* $A\in\mathcal{A}$, *we have a* $(s_A,k,1)$*-GDD of type* $\{s(x)\colon x\in A\}$. *Then there exists a* $(s_X,k,1)$*-HPMD of type* $\{s_G\colon G\in\mathcal{G}\}$.

Theorem 3.3. *Suppose* $(X,\mathcal{G},\mathcal{A})$ *is a GDD with* $\lambda=1$ *and for every block* $A\in\mathcal{A}$, *suppose that we have a* $(s_A,t_A,k,1)$*-IHPMD of type* $\{(s(x),t(x))\colon x\in A\}$. *Then there exists a* $(s_X,t_X,k,1)$*-IHPMD of type* $\{(s_G,t_G)\colon G\in\mathcal{G}\}$.

Table 1

$v-n$	n	Exist or not	From
8	2	Exist	[3]
$12, 16, \ldots, 32$	2	Exist	[14]
$12, 16, \ldots, 32$	$4 \leqslant n \leqslant (v-n-2)/2$	Exist	[14]
5	2	No	[3]
9	2	Exist	[3]
9	4	Exist	[12]
13	6	Exist	[3]
13	2	Exist	Lemma 2.5
13	4	Exist	[12]
17	8	Exist	[3]
17	2	Unknown	
17	4, 6	Exist	Theorem 2.4
21	2	Exist	Lemma 2.1
21	4, 6, 8, 10	Exist	Theorem 2.4
25	12	Exist	[3]
25	2	Unknown	
25	4, 6, 8, 10	Exist	Theorem 2.4
29	2, 14	Exist	[3]
29	4, 6, 8, 10, 12	Exist	Theorem 2.4
11, 15, 19	5	Exist	[12]
19	9	Exist	[12]
23	9, 11	Exist	[12]
23	5	Exist	Theorem 2.4
27	13	Exist	[12]
27	5, 9, 11	Exist	Theorem 2.4
31	5, 13, 15	Exist	[12]
31	9, 11	Exist	Theorem 2.4

Theorem 3.4. *Suppose $(X,\mathscr{G},\mathscr{A})$ is a $(v,k,1)$-HPMD, and for every block $A\in\mathscr{A}$, suppose that we have a $(s_A,t_A,k,1)$-IGDD of type $\{(s(x),t(x))\colon x\in A\}$. Then there exists a $(s_X,t_X,k,1)$-IHPMD of type $\{(s_G,t_G)\colon G\in\mathscr{G}\}$.*

Theorem 3.5. *Suppose $(X,Y,\mathscr{G},\mathscr{A})$ is a IGDD with $\lambda=1$ and for every block $A\in\mathscr{A}$, we have a $(s_A,k,1)$-HPMD of type $\{s(x)\colon x\in A\}$. Then there exists a $(s_X,s_Y,k,1)$-IHPMD of type $\{(s_G,s_{G\cap Y})\colon G\in\mathscr{G}\}$.*

Theorem 3.6. *Suppose $(X,Y,\mathscr{G},\mathscr{A})$ is a $(v,u,k,1)$-IHPMD, and for every block $A\in\mathscr{A}$, we have a $(s_A,k,1)$-GDD of type $\{s(x)\colon x\in A\}$. Then there exists a $(s_X,s_Y,k,1)$-IHPMD of type $\{(s_G,s_{G\cap Y})\colon G\in\mathscr{G}\}$.*

Theorem 3.7 (Filling in groups). *Suppose that the following designs exist*:

(i) *a $(v,k,1)$-HPMD of type $(g_1,g_2,\ldots,g_h)$*;
(ii) *a $(g_i+a,a,k,1)$-IPMD, for $1\leqslant i\leqslant h-1$.*

Then there exists a $(v+a,g_h+a,k,1)$-IPMD.

The following theorem provides a way to obtain an IPMD from an IHPMD and some $(v, w_1, w_2, w, k, \lambda)$-$\diamondsuit$-IPMD, which is a variation of construction 3.3 in [11].

Theorem 3.8 (Filling in groups). *Suppose that the following designs exist*:

(i) *a* $(v, u, k, 1)$-*IHPMD of type* $((g_1, u_1), (g_2, u_2), \ldots, (g_h, u_h))$;
(ii) *a* $(g_i + b, u_i + a, b, a, k, 1)$-$\diamondsuit$-*IPMD*, *for* $1 \leqslant i \leqslant h - 1$; *and*
(iii) *a* $(g_h + b, u_h + a, k, 1)$-*IPMD*.

Then there exists a $(v + b, u + a, k, 1)$-*IPMD*.

The following theorem provides a way to obtain a $\diamondsuit$-IPMD from an HPMD.

Theorem 3.9 (Filling in groups). *Suppose that the following designs exist*:

(i) *a* $(v, k, 1)$-*HPMD of type* $(g_1, g_2, \ldots, g_h)$;
(ii) *a* $(g_i + a, a, k, 1)$-*IPMD for* $3 \leqslant i \leqslant h$.

Then there exists a $(v + a, g_1 + a, g_2 + a, a, k, 1)$-$\diamondsuit$-*IPMD*.

The following theorem provides a way to obtain a 4-IHPMD by applying Theorem 3.3.

Theorem 3.10. *There exists an* 4-*IHPMD of type* $(3m, m)^4 (r, u)^1$ *for* $m \geqslant 4$ *and* $m \neq 6, 10$, *where* $r - u = 0, 3, 6, \ldots, 3m$ *when* $u = 0$; $r - u = 2, 3, 5, 6, \ldots, 3m - 1, 3m$ *when* $u = 1$; *and* $2u \leqslant r - u \leqslant 3m$ *when* $2 \leqslant u \leqslant m$.

Proof. First, we start with a $(5, 4, 1)$-PMD and replace each block with a $\{4\}$-ITD of type $(3, 1)^4$ to obtain an IHPMD of type $(3, 1)^5$ by Theorem 3.4. Combining this with Lemma 2.2, we have IHPMDs of type $(3, 1)^4 (x, y)^1$ for $(x, y) \in \{(0, 0), (3, 0), (3, 1), (4, 1)\}$. It is well known that there exists a TD$(5, m)$ for $m \geqslant 4$ and $m \neq 6, 10$ (see [7]). Let X be the points set of the TD. Partition one group of the TD into $Y_1 \cup Y_2$ such that $|Y_1| = u$, $|Y_2| = m - u$. Take $(s(x), t(x)) = (3, 1)$ or $(4, 1)$ if $x \in Y_1$, $(s(x), t(x)) = (3, 0)$ or $(0, 0)$ if $x \in Y_2$, $(s(x), t(x)) = (3, 1)$ otherwise. We apply Theorem 3.3 to obtain an IHPMD of type $(3m, m)^4 (r, u)^1$. $\square$

We need the following lemmas for later use.

Lemma 3.11. *Let* $x_i, y_i, z_i \geqslant 0$, $1 \leqslant i \leqslant m$, $x = x_1 + x_2 + \cdots + x_m$, $y = y_1 + y_2 + \cdots + y_m$, $z = z_1 + z_2 + \cdots + z_m$.

Suppose that there exists a TD$(h + 3, m)$ (*if* $z = 0$, *only need a* TD$(h + 2, m)$). *Suppose that there exist* 4-*HPMDs of type* $1^h x_i^1 y_j^1 z_k^1$, $1 \leqslant i, j, k \leqslant m$.

Then there exists a 4-*HPMD of type* $m^h x^1 y^1 z^1$.

Proof. Apply Theorem 3.1 with $s(x) = x_i$, if $x = g_{1i} \in G_1$, $s(x) = y_i$, if $x = g_{2i} \in G_2$, $s(x) = z_i$, if $x = g_{3i} \in G_3$, and $s(x) = 1$ otherwise. $\square$

Lemma 3.12. *Let* $0 \leqslant t_1, t_2 \leqslant m$, $0 \leqslant s \leqslant 5m$. *Suppose that there exists a* TD$(13, m)$ (*if* $s=0$, *only need a* TD$(12, m)$). *Then there exists a* 4-*HPMD of type* $m^{10}(m+4t_1)^1 (m+4t_2)^1 s^1$.

Proof. From Theorem A.2 we have 4-HPMDs of type $5^4 1^1$, so we have a 4-HPMD of type $1^{11}5^2$; from Lemma 2.3 we have a 4-HPMD of type $1^{10}5^2$; from [12,13] we have $1^{11}5^1$, $1^{12}5^1$; and from Theorem 1.5 we have a 4-HPMD of type 5^5, so we have a 4-HPMD of type $1^{10}5^3$. Hence, we have 4-HPMDs of type $1^{10}x^1y^1z^1$, $x = 1, 5, y = 1, 5,\ z = 0, 1, 5$. Therefore, we have the required 4-HPMD by applying Lemma 3.11. □

Lemma 3.13. *Let* $0 \leqslant t \leqslant m$, $2 \leqslant s \leqslant 6m$. *Suppose that there exists a* TD$(6, m)$ (*if* s=0 *or* t=0, *only need a* TD$(5, m)$). *Then there exists a* 4-*HPMD of type* $(4m)^4(4t)^1 s^1$ *where* $0 \leqslant t \leqslant m$, $2 \leqslant s \leqslant 6m$.

Proof. From Theorems A.2 and A.3 we have 4-HPMDs of type 4^4x^1 and 4^5y^1 for $x, y = 0, 2, 3, 4, 5, 6$. Therefore, we have the required 4-HPMD by applying Lemma 3.11. □

Combining Theorems 3.9 and 1.5 we have

Lemma 3.14. *Let* $m \equiv 0 \pmod 4$ *and* $0 \leqslant a \leqslant (m-2)/2$. *If there exists a* $(m+a, a, 4, 1)$-*IPMD, then there exists a* $(4m+a, m+a, m+a, a, 4, 1)$-$\diamondsuit$-*IPMD.*

Lemma 3.15. *Let* $0 \leqslant u \leqslant m$, $s \in \{m, m+2, \ldots, 5m\}$. *If there is a* TD$(13, m)$ (*only need a* TD$(12, m)$ *if* $s=0$), *then there exists a* $(v, 4, 1)$-*HPMD of type* $m^{11}u^1s^1$, $0 \leqslant u \leqslant m$, $s = m, m+2, \ldots, 5m$.

Proof. By applying Lemma 3.11 with 4-HPMDs of type $1^{11}x^1y^1$, $x = 0, 1$, $y = 1, 3, 5$, which comes from $(v, n, 4, 1)$-IPMD for $v - n = 11, 12$, $n = 1, 3, 5$, we have the required HPMD. □

4. (*v*, *n*, 4, 1)-IPMD

In this section we will show that the necessary conditions for the existence of $(v, n, 4, 1)$-IPMD are sufficient except for $(v-n, n) \in \{(4, 0), (8, 0), (3, 1), (7, 1)\}$ and possible excepting $(v-n, n) \in \{(17, 2), (25, 2)\}$.

Let $E = \{(4, 0), (8, 0), (3, 1), (7, 1)\}$, from Theorem 1.1 we have

Lemma 4.1. *There exists a* $(v, n, 4, 1)$-*IPMD for admissible* $(v-n, n)$ *with* $n = 0, 1$ *and except* $(v-n, n) \in E$.

Lemma 4.2 (Bennett et al. [3,4]). *There exists a* $(v, n, 4, 1)$-*IPMD for admissible* $(v-n, n)$ *with* $n = 3, 7$.

Lemma 4.3 (Xuebin Zhang [12]). *If* $2n+1$ *is prime power, then there exists a* $(3n+1,n,4,1)$*-IPMD for odd* n.

Lemma 4.4 (Bennett et al. [3]). *If* $2n+1$ *is prime power, then there exists a* $(3n+1,n,4,1)$*-IPMD for even* n.

Combining Theorem 2.6 and Lemma 4.1 we have

Theorem 4.5. *There exists a* $(v,n,4,1)$*-IPMD for admissible* $(v-n,n)$ *with* $v-n \leqslant 32$, *except* $(v-n,n)\in E$, *and possibly except* $(v-n,n)\in\{(17,2),(25,2)\}$.

It is clear that there exist a $(v,u,4,1)$-IPMD and a $(u,n,4,1)$-IPMD implies there exists a $(v,n,4,1)$-IPMD. So we have the following lemma.

Lemma 4.6. *If there exist the following* $(v,n,4,1)$*-IPMDs*:

(i) $v-n=8k-4,8k,\ k\leqslant n\leqslant(v-n-2)/2$, *for* $k\in N$ *and* $n\geqslant 4,\ v-n\leqslant s$;
(ii) $v-n=8k-11,8k-7,8k-3,8k+1,\ k\leqslant$ *even* $n\leqslant(v-n-2)/2$, *for even* k *and* $n\geqslant 4,\ v-n\leqslant s$;
(iii) $v-n=8k-9,8k-5,8k-1,8k+3,\ k\leqslant$ *odd* $n\leqslant(v-n-2)/2$, *for odd* k, $n\geqslant 5$, $v-n\leqslant s$.

Then there exists a $(v,n,4,1)$*-IPMD for admissible* $(v-n,n)$ *with* $n\geqslant 4, v-n\leqslant s$.

Theorem 4.7. *There exists a* $(v,n,4,1)$*-IPMD for admissible* $(v-n,n)\in\{(36,5),(36,12-17),(40,9-19),(44,6-21),(52,6-25),(56,7-21),(68,9-33),(76,10-37),(45,6-10),(77,20-28),(77,10-16),(35,5),(119,17),(119,21),(119,25)\}$.

Proof. To obtain the required results, we apply Theorem 3.7 with the following 4-HPMDs and a:

(i) 4^{10} and $a=1$;
(ii) 12^4 and $a=0$–5;
(iii) 8^5x^1, $x=8,10,\ldots,16$ and $a=1$–3;
(iv) 11^4x^1, $x=5$–16 and $a=1,3,5$;
(v) 13^4x^1, $x=6$–19 and $a=0,2,4,6$;
(vi) 8^7x^1, $x=6,8,\ldots,18$ and $a=1$–3;
(vii) 17^4x^1, $x=9$–25 and $a=0,4,6,8$;
(viii) 19^4x^1, $x=9$–28 and $a=1,3,\ldots,9$;
(ix) 9^56^1 and $a=0,2,4$;
(x) 19^420^1 and $a=1,3,\ldots,9$;
(xi) 11^8 and $a=1,3,5$;
(xii) 7^{12} and $a=3$;
(xiii) 5^8 and $a=0$;
(xiv) 17^8 and $a=0,4,8$.

Here the 4-HPMDs come from Theorems 1.5, A.2, and A.3. □

Lemma 4.8. *There exist* 4-*HPMDs of the following types*:

(i) 8^5x^1, $x = 4$–12;
(ii) 8^9x^1, $x = 4$–12;
(iii) 8^4x^1, $x = 4$–12;
(iv) 16^4x^1, $x = 0, 2$–24;
(v) 20^4x^1, $x = 0, 2$–30;
(vi) 28^4x^1, $x = 0, 2$–42;
(vii) 27^4x^1, $x = 18$–36;
(viii) 36^4x^1, $x = 0, 2$–54;
(ix) 24^4x^1, $x = 12$–36;
(x) 20^5x^1, $x = 0, 2$–$38, 40$;
(xi) 21^5x^1, $x = 0, 4, 6, \ldots, 42$;
(xii) $9^6 27^1 x^1$, $x = 9, 11, \ldots, 27$;
(xiii) $13^6 39^1 x^1$, $x = 13, 15, \ldots, 39$;
(xiv) 8^8x^1, $x = 8$–24;
(xv) 9^8x^1, $x = 9$–27;
(xvi) 13^8x^1, $x = 13$–39;
(xvii) 9^9x^1, $x = 0, 2, \ldots, 36$;
(xviii) 11^9x^1, $x = 0, 2, \ldots, 44$;
(xix) 13^9x^1, $x = 0, 2, \ldots, 52$;
(xx) $13^{13}x^1$, $x = 0, 2, \ldots, 78$;
(xxi) $11^{10}x^1y^1$, $x, y = 11, 15, \ldots, 55$;
(xxii) $13^{10}x^1y^1$, $x, y = 13, 17, \ldots, 65$.

Proof. To obtain the required results, we apply Theorem 3.1 with the following GDDs and 4-HPMDs:

(i) 5-GDD of type 4^6 and 2^4x^1, $x = 1$–3;
(ii) 5-GDD of type 4^{10} and 2^4x^1, $x = 1$–3;
(iii) TD(5, 4) and 2^4x^1, $x = 1$–3;
(iv) TD(5, 4) and 4^4x^1, $x = 0, 2$–6;
(v) TD(5, 5) and 4^4x^1, $x = 0, 2$–6;
(vi) TD(5, 7) and 4^4x^1, $x = 0, 2$–6;
(vii) TD(5, 9) and 3^4x^1, $x = 2$–4;
(viii) TD(5, 9) and 4^4x^1, $x = 0, 2$–6;
(ix) TD(5, 12) and 2^4x^1, $x = 1$–3;
(x) TD(6, 5) and 4^5x^1, $x = 0, 2$–$6, 8$;
(xi) TD(6, 7) and 3^5x^1, $x = 0, 4, 6$;
(xii) TD(8, 9) and $1^6 3^1 x^1$, $x = 1, 3$;
(xiii) TD(8, 13) and $1^6 3^1 x^1$, $x = 1, 3$;
(xiv) TD(9, 8) and 1^8x^1, $x = 1$–3;
(xv) TD(9, 9) and 1^8x^1, $x = 1$–3;
(xvi) TD(9, 13) and 1^8x^1, $x = 1$–3;
(xvii) TD(10, 9) and 1^9x^1, $x = 0, 2, 4$;

(xviii) TD(10, 11) and 1^9x^1, $x = 0, 2, 4$;
(xix) TD(10, 13) and 1^9x^1, $x = 0, 2, 4$;
(xx) TD(14, 13) and $1^{13}x^1$, $x = 0, 4, 6$;
(xxi) TD(12, 11) and $1^{10}x^1y^1$, $x, y = 1, 5$;
(xxii) TD(12, 13) and $1^{10}x^1y^1$, $x, y = 1, 5$.

Here the 4-HPMDs come from Theorems 4.5, A.2 and A.3 and Lemmas 3.12 and 2.3. □

Lemma 4.9. *There exist* 4-*HPMDs of the following types*:

(i) 9^4x^1, $x = 6, 9, 12$;
(ii) 12^4x^1, $x = 0, 6, 9, \ldots, 18$;
(iii) 12^5x^1, $x = 0, 6, 9, \ldots, 18, 24$;
(iv) 9^5x^1, $x = 0, 12, 18$.

Proof. To obtain the required results, we apply Theorem 3.2 with the following 4-HPMDs and TD(4, 3):

(i) 3^4x^1, $x = 2$–4;
(ii) 4^4x^1, $x = 0, 2$–6;
(iii) 4^5x^1, $x = 0, 2$–$6, 8$;
(iv) 3^5x^1, $x = 0, 4, 6$. □

Applying Theorem 3.7 with the HPMDs in Lemmas 4.8 and 4.9, and Theorems 4.5 and A.3, we have

Theorem 4.10. *There exists a* $(v, n, 4, 1)$-*IPMD for admissible* $(v-n, n) \in \{(36, 6), (36, 8$–$12), (40, 5$–$15), (73, 10), (48, 0$–$23), (60, 0$–$29), (64, 0$–$31), (72, 9$–$31), (100, 0$–$49), (104, 13$–$45), (108, 19$–$49), (45, 12$–$22), (65, 24$–$30), (69, 16$–$22), (81, 0$–$40), (105, 0$–$52), (117, 0$–$58), (121 + 4k, 12$–$60), 0, 1, \ldots, 11, (169, 0$–$84), (35, 9$–$11), (59, 17$–$21), (63, 17$–$23), (63, 27$–$31), (67, 9$–$11), (71, 27), (79, 21$–$29), (83, 21$–$29), (87, 25$–$31), (91, 25$–$33), (95, 25$–$35), (99, 1$–$49), (103, 29$–$37), (107, 25$–$35), (111, 29$–$41), (111, 43$–$45), (139, 37$–$51), (143 + 4k, 13$–$71), k = 0, 1, \ldots, 13\}$.

Applying Theorem 3.5 with ITDs $((4, 1)^5, (5, 1)^5, (5, 1)^6, (7, 1)^5, (8, 1)^5, (8, 2)^5, (9, 1)^5, (9, 1)^7, (9, 2)^5, (10, 2)^5, (10, 2)^6, (13, 3)^5)$ from Theorem 3.10 in [7], and HPMDs $(2^4x^1, 2^5x^1, 2^6x^1, 3^4x^1, 4^4x^1, 4^5x^1)$ in Theorems A.2 and A.3, we have the following lemma.

Lemma 4.11. *There exist* 4-*IHPMDs of the following types*:

(i) $\{(12 + y_i, y_i)\colon i = 1$–$5\}$, $y_i = 0, 2$–6;
(ii) $(12 + 3, 3)^4(x, y)^1$, $x - y = 8$–16, $y = 2$–4;
(iii) $(8 + 2, 2)^4(x, y)^1$, $x - y = 4$–12, $y = 1$–3;
(iv) $(16 + 4, 4)^4(12 + y, y)$, $y = 0, 2$–6;

(v) $\{(16+y_i,y_i)\colon i=1\text{–}4\}(x,y)$, $y_i=0,4$, $x-y=16$, $y=0,2\text{–}6$;
(vi) $\{(16+y_i,y_i)\colon i=1\text{–}5\}(x,y)$, $y_i=0,4$, $x-y=2\text{–}24$, $y=0,2\text{–}6,8$;
(vii) $\{(16+y_i,y_i)\colon i=1\text{–}5\}(x,y)$, $y_i=0,2\text{–}6$, $x-y=0,4,8,12,16$, $y=0,2\text{–}6,8$;
(viii) $\{(8+y_i,y_i)\colon i=1\text{–}6\}$, $y_i=0,2,4$;
(ix) $(12+2,2)^4(x,y)^1$, $x-y=6\text{–}18$, $y=1\text{–}3$;
(x) $(12+2,2)^5(x,y)^1$, $x-y=0,2,\ldots,24$, $y=0,2,4$;
(xi) $\{(14+y_i,y_i)\colon i=1\text{–}5\}$, $y_i=1\text{–}3$;
(xii) $(32,8)^4(x,y)$, $x-y=7\text{–}36$, $y=0,2\text{–}12$;
(xiii) $(30,6)^4(x,y)$, $x-y=16\text{–}32$, $y=4\text{–}8$;
(xiv) $(16+2,2)^4(x,y)$, $x-y=8\text{–}24$, $y=1\text{–}3$;
(xv) $\{(16+y_i,y_i)\colon i=1\text{–}5\}$, $y_i=0,2\text{–}6$;
(xvi) $\{(14+y_i,y_i)\colon i=1\text{–}5\}$, $y_i=2\text{–}6$;
(xvii) $\{(16+y_i,y_i)\colon i=1\text{–}6\}(x,y)$, $y_i=0,2,4$, $x-y=0,2,\ldots,16$, $y=0,2\text{–}4$;
(xviii) $\{(32+y_i,y_i)\colon i=1\text{–}5\}(x,y),y$, $y_i=0,2\text{–}12$, $x-y=0,4,\ldots,32$;
(xix) $\{(32+y_i,y_i)\colon i=1\text{–}6\}$, $y_i=0,2\text{–}14,16$;
(xx) $\{(20+y_i,y_i)\colon i=1\text{–}5\}$, $y_i=3\text{–}9$.

Applying Theorem 3.8 with the IHPMDs in Lemma 4.11, we have

Theorem 4.12. *There exists a $(v,n,4,1)$-IPMD for admissible $(v-n,n)$ with $(v-n,n)\in$ $\{(104,0\text{–}31)$, $(108,1\text{–}27)$, $(49,0\text{–}24)$, $(61,4\text{–}30)$, $(65,8\text{–}22)$, $(85,32)$, $(85,0\text{–}30)$, $(89,0\text{–}34)$, $(93,0\text{–}36)$, $(97,4\text{–}38)$, $(101,20\text{–}40)$, $(173+4k,4\text{–}66)$, $k=0,1,\ldots,4$, $(193,4\text{–}96)$, $(39,11)$, $(55,9\text{–}17)$, $(59,9\text{–}17)$, $(63,9\text{–}17)$, $(71,15\text{–}25)$, $(75,5\text{–}27)$, $(79,9\text{–}23)$, $(83,1\text{–}27)$, $(111,3\text{–}27)$, $(115,15\text{–}43)$, $(119,29)$, $(119,33\text{–}43)$, $(123,33\text{–}45)$, $(127,33\text{–}47)$, $(131,33\text{–}47)\}$.*

Applying Theorem 3.8 with IHPMD in Theorem 3.10 and $\diamondsuit$-IPMDs in Lemma 2.3 with $b-a=m$, we have

Theorem 4.13. *There exists a $(v,n,4,1)$-IPMD for admissible $(v-n,n)$ in Table* 2.

Applying Theorem 3.8 with IHPMD in Theorem 3.10 with $a=0$, $b=1$, we have

Theorem 4.14. *There exists a $(v,n,4,1)$-IPMD for admissible $(v-n,n)$ in Table* 3.

Applying Theorem 3.7 with HPMD in Lemma 3.13, we have

Theorem 4.15. *There exists a $(v,n,4,1)$-IPMD for admissible $(v-n,n)$ in Table* 4.

Combining the results in this section and Theorem A.1, we have

Theorem 4.16. *There exists a $(v,n,4,1)$-IPMD for admissible $(v-n,n)$ with $v-n\leqslant 193$ and $n\geqslant 3$.*

Lemma 4.17. *If there exists a $(v,n,4,1)$-IPMD for admissible $(v-n,n)$ and $7\leqslant v-n\leqslant 3m+1$ and $3\leqslant n\leqslant m$, then there exists a $(v,n,4,1)$-IPMD for admissible $(v-n,n)$ and $8m+7\leqslant v-n\leqslant 11m+1$ and $4m+3\leqslant n\leqslant 5m$, where $m\geqslant 11$.*

Table 2

m	u	$s=r-u+m$	$t=u+a$	$v-n=8m+s$	$n=4m+t$
4	1	9	2	41	18
4	2–4	7–16	3–5	39–48	19–21
5	1	11	1	51	21
5	2	16	2	56	22
5	3	11–20	3	51–60	23
5	4	13–20	4	53–60	24
5	3–5	11–20	5	51–60	25–27
7	1	21	2, 4	77	30, 32
7	2–7	11–28	3–10	67–84	31–38
8	0	23	1	87	33
8	2–8	12–32	3–11	76–96	35–43
9	0	33	0	105	36
9	1	27, 35	1	99, 107	37, 39, 41
9	2	13–36	4, 6	85–108	40, 42
9	3–9	15–36	3–13	87–108	39–49
11	1	37–44	2	125–132	46
11	1	13–17	6	101–105	50
11	2–11	15–44	3–16	103–132	47–60
12	0	45	0	141	48
12	1	39, 44–48	1	135, 140–144	49
12	2	36–48	2	132–144	50
12	2–12	16–48	3–17	112–144	51–65
16	2	40–64	2–9	168–192	66
16	3–16	22–64	3–23	150–192	67–87

Proof. Apply Theorem 3.8 with 4-IHPMD in Theorem 3.10 and $a=0$, $b=1$, $3 \leqslant n \leqslant m$. Here the required $\diamondsuit$-IPMD comes from Lemma 1.6. $\square$

Lemma 4.18. *If there exists a $(v,n,4,1)$-IPMD for admissible $(v-n,n)$ and $m+6 \leqslant v-n \leqslant 4m$ and $3 \leqslant n \leqslant m$, then there exists a $(v,n,4,1)$-IPMD for admissible $(v-n,n)$ and $9m+6 \leqslant v-n \leqslant 12m$ and $4m+3 \leqslant n \leqslant 5m+(m-2)/2$, where $m \geqslant 12$, $m \equiv 0 \pmod 4$.*

Proof. Apply Theorem 3.8 with 4-IHPMD in Theorem 3.10 and $b-a=m$, $0 \leqslant a \leqslant (m-2)/2, 3 \leqslant n \leqslant m$. Here the required $\diamondsuit$-IPMD comes from Lemma 3.14. $\square$

Applying Theorem 3.7 with HPMD in Lemma 3.15, we have the following four lemmas.

Lemma 4.19. *There exists a $(v,n,4,1)$-IPMD for admissible $(v-n,n)$ and $v-n \equiv 0 \pmod 4$, $11m+1 \leqslant v-n \leqslant 12m$ and $m \leqslant$ odd $n \leqslant 5m$, where $m=4k+1$, $N(m) \geqslant 11$.*

Lemma 4.20. *There exists a $(v,n,4,1)$-IPMD for admissible $(v-n,n)$ and $v-n \equiv 3 \pmod 4$, $11m+12 \leqslant v-n \leqslant (12m-1)$ and $m \leqslant$ odd $n \leqslant 5m$, where $m=4k+1$, $N(m) \geqslant 11$.*

Table 3

m	u	$s = r - u + 1$	$t = u$	$v - n = 8m + s$	$n = 4m + t$
5	1	15	1	55	21
5	3–5	7–16	3–5	47–56	23–25
7	0	1	0	57	28
7	1	15, 19	1	71, 75	29
7	3–7	7–22	3–7	63–78	31–35
8	2	21, 24	2	85, 88	34
8	3–8	7–25	3–8	71–89	35–40
9	1	15, 19, 27	1	87, 91, 99	37
9	3–9	7–28	3–9	79–100	39–45
11	0	28	0	116	44
11	1	24–31	1	112–119	45
11	2	21–24	2	109–112	46
11	2	28–32	2	116–120	46
11	3–11	7–34	3–11	95–122	47–55
12	3–12	7–37	3–12	103–133	51–60
14	0	37, 40	0	149, 152	56
14	1	36–43	1	148–155	57
14	2	33–43	2	145–155	58
14	3–14	7–43	3–14	119–155	59–70
15	1	36–43	1	156–163	61
15	2	36–45	2	156–165	62
15	3–15	7–45	3–15	127–165	63–75
18	3–18	7–55	3–18	151–199	75–90
19	3–19	7–58	3–19	159–210	79–95

Lemma 4.21. *There exists a $(v, n, 4, 1)$-IPMD for admissible $(v - n, n)$ and $v - n \equiv 0 \pmod 4$, $11m + 7 \leqslant v - n \leqslant 12m$ and $m + 3 \leqslant$ even $n \leqslant 5m + 3$, where $m = 4k + 3$, $N(m) \geqslant 11$.*

Lemma 4.22. *There exists a $(v, n, 4, 1)$-IPMD for admissible $(v - n, n)$ and $v - n \equiv 1 \pmod 4$, $11m \leqslant v - n \leqslant (12m - 3)$ and $m + 1 \leqslant$ even $n \leqslant 5m + 1$, where $m = 4k + 3$, $N(m) \geqslant 11$.*

Applying Theorem 3.7 with HPMD in Lemma 3.12, and $a = 0, 4, \ldots, (m - 1)/2$ for $m = 4k + 1$ and $a = 1, 3, \ldots, (m - 1)/2$ for $m = 4k + 3$, we have the following four lemmas.

Lemma 4.23. *If there exists a $(v, n, 4, 1)$-IPMD for admissible $(v - n, n)$ and $m \leqslant v - n \leqslant 3m + 1$ and $n = 0, 4, 6, \ldots, (m - 1)/2$, then there exists a $(v, n, 4, 1)$-IPMD for admissible $(v - n, n)$ and $v - n \equiv 0 \pmod 4$, $12m \leqslant v - n \leqslant 15m + 1$ and $0 \leqslant n \leqslant 5m + (m - 1)/2$, where $m = 4k + 1$, $N(m) \geqslant 12$.*

Lemma 4.24. *If there exists a $(v, n, 4, 1)$-IPMD for admissible $(v - n, n)$ and $m \leqslant v - n \leqslant 3m$ and $n = 0, 4, 6, \ldots, (m - 1)/2$, then there exists a $(v, n, 4, 1)$-IPMD for admissible $(v - n, n)$ and $v - n \equiv 3 \pmod 4$, $12m + 3 \leqslant v - n \leqslant 15m$ and $m + 4 \leqslant$ odd $n \leqslant 5m + (m - 1)/2$, where $m = 4k + 1$, $N(m) \geqslant 12$.*

Table 4

m	$t=0-m$	a	$v-n=16m+4t$	$n=2+a-6m+a$
5	0	0–9	80	2–39
5	1	1	84	3–31
5	2	1–3	88	3–33
5	3	0–5	92	2–35
5	4	0–7	96	2–37
7	0	0–13	112	2–55
7	1	1	116	3–43
7	2	1–3	120	3–45
7	3	0–5	124	2–47
7	5	0–9	132	2–51
8	0	0–15	128	2–63
8	2	1–3	136	3–51
8	3	0–5	140	2–53
8	5	0–9	148	3–57
8	7	0–13	156	2–61
9	0	0–17	144	2–71
9	2	1–3	152	2–57
9	4	0–7	160	2–61
9	5	0–9	164	2–63
9	6	0–11	168	2–65
9	7	0–13	172	2–67
9	8	0–15	176	2–69
9	9	0–17	180	2–71
11	2	1–3	184	3–69
11	3	0–5	188	2–71
11	4	0–7	192	2–73
m	s	a	$v-n=16m+s$	$n=a-4m+a$
5	11	$1,3,5$	91	1–25
5	15	$1,3,5,7$	95	1–27
5	$19,23,27$	$1,3,\ldots,9$	$99,103,107$	1–29
7	7	3	119	$3,7,\ldots,31$
7	11	$1,3,5$	123	1–33
7	15	$1,3,5,7$	127	1–35
7	19	$1,3,\ldots,9$	131	1–37
7	23	$1,3,\ldots,11$	135	1–39
7	27	$1,3,\ldots,13$	139	1–41
5	$17,21,25,29$	$4,6,8$	$97,101,105,109$	4–28

Lemma 4.25. *If there exists a $(v,n,4,1)$-IPMD for admissible $(v-n,n)$ and $m \leqslant v-n \leqslant 3m-1$ and $n=1,3,\ldots,(m-1)/2$, then there exists a $(v,n,4,1)$-IPMD for admissible $(v-n,n)$ and $v-n \equiv 0 \pmod 4$, $12m \leqslant v-n \leqslant 15m-1$ and $1 \leqslant n \leqslant 5m+(m-1)/2$, where $m=4k+3$, $N(m) \geqslant 12$.*

Lemma 4.26. *If there exists a $(v,n,4,1)$-IPMD for admissible $(v-n,n)$ and $m \leqslant v-n \leqslant 3m$ and $n=1,3,\ldots,(m-1)/2$, then there exists a $(v,n,4,1)$-IPMD for admissible $(v-n,n)$ and $v-n \equiv 1 \pmod 4$, $12m+1 \leqslant v-n \leqslant 15m$ and $m+1 \leqslant$ even $n \leqslant 5m+(m-1)/2$, where $m=4k+3$, $N(m) \geqslant 12$.*

Lemma 4.27. *There exists a $(v,n,4,1)$-IPMD for the following admissible $(v-n,n)$*:

(i) $v-n=465,469,\ 4\leqslant n\leqslant 174$;
(ii) $v-n=255,259,\ 87\leqslant n\leqslant 101$;
(iii) $v-n=263,267,\ldots,283,\ 69\leqslant n\leqslant 97$;
(iv) $v-n=435,\ 113\leqslant n\leqslant 161$;
(v) $v-n=193,197,\ldots,205,\ 20\leqslant n\leqslant 86$;
(vi) $v-n=195,199,\ldots,215,\ 17\leqslant n\leqslant 87$.

Proof. (i) By applying Theorem 3.1 with TD(18, 29) and 4-HPMD of type $1^{16}x^1y^1$, $x=0,1,y=0,4,6$, we have HPMD of type $29^{16}n^1s^1$ for $n=1,5$ and $s=4,6,\ldots,174$. Hence we have the required 4-IPMDs.

(ii) By applying Theorem 3.1 with TD(8, 29) and 4-HPMD of type $1^6x^1y^1$, $x=1,3,y=3$, we have HPMD of type $29^6n^1s^1$ for $n=81,85$ and $s=87$. Taking $a=0,2,\ldots,14$, we have the required 4-IPMDs.

(iii) By applying Theorem 3.1 with TD(5, 17) and 4-HPMD of type 4^4x^1, $x=0,2,3,4,5,6$, we have HPMD of type 68^4s^1 for $s=59,63,\ldots,79$. Taking $a=1,3,\ldots,29$, we have the required 4-IPMDs.

(iv) By applying Theorem 3.1 with TD(5, 28) and 4-HPMD of type 4^4x^1, $x=0,2,3,4,5,6$, we have HPMD of type $112^4 99^1$. Taking $a=1,3,\ldots,49$, we have the required 4-IPMD.

(v) By applying Lemma 3.12 with $m=16$, we have HPMD of type $16^{11}(16+4t)^1s^1$ for $t=0,1,\ldots,16$, $s=17,21,25,29$. Taking $a=4,6$, we have the required 4-IPMD.

(vi) By applying Lemma 3.12 with $m=16$, we have HPMD of type $16^{11}(16+4t)^1s^1$ for $t=0,1,\ldots,16$, $s=19,23,\ldots,39$. Taking $a=1,3,5,7$, we have the required 4-IPMD. □

Theorem 4.28. *There exists a $(v,n,4,1)$-IPMD for admissible $(v-n,n)$ in Table* 5.

From Lemma 4.18 we have

Lemma 4.29. *Let $m_k=56+4k$, $k=0,1,2,\ldots$. If there exists a $(v,n,4,1)$-IPMD for admissible $(v-n,n)$ and $m_k+6\leqslant v-n\leqslant 4m_k$ and $3\leqslant n\leqslant m_k$, then there exists a $(v,n,4,1)$-IPMD for admissible $(v-n,n)$ and $672=12m_0\leqslant v-n$ and $0.362=243/672=4m_0+19/12m_0\leqslant n/(v-n)$.*

Let $\{m_k\}=53,61,73,81,89,97,101,109,113,121,125,137$. From Lemmas 4.23 and 4.24 we have

Lemma 4.30. *If there exists a $(v,n,4,1)$-IPMD for admissible $(v-n,n)$ and $m_k\leqslant v-n\leqslant 3m_k+1$ and $n=0,4,6,\ldots,(m_k-1)/2$, then there exists a $(v,n,4,1)$-IPMD for admissible $(v-n,n)$ and $v-n\equiv 0\pmod 4$, $800=15m_0+5\leqslant v-n\leqslant 15m_{11}+5=2060$ and $n/(v-n)\leqslant(11m_k-1)/(30m_k+2)(\geqslant 0.365)$.*

Lemma 4.31. *If there exists a $(v,n,4,1)$-IPMD for admissible $(v-n,n)$ and $m_k\leqslant v-n\leqslant 3m_k$ and $n=0,4,6,\ldots,(m_k-1)/2$, then there exists a $(v,n,4,1)$-IPMD for ad-*

Table 5

$v-n$	m	Lemmas 4.23–4.26	Lemmas 4.19–4.22	Lemma 4.27	Lemmas 4.18
188–204	17		17–85		86–
208–228	17	0–93			94–
232–276	19	1–104			105–
280–324	23	1–126			127–
328–372	27	1–148			149–
376–444	31	1–170			171–
448–516	37	0–203			204–
520–564	43	1–236			237–
568–636	47	1–258			259–
640–732	53	0–291			292–
736–916	61	0–335			336–
193–205	16			20–86	88–
209–225	19		20–96		98–
229–273	19	20–104			106–
277–321	23	24–126			128–
325–373	27	28–148			150–
377–461	31	32–170			172–
465–469	29			4–174	176–
473–513	43		44–216		218–
517–561	43	44–236			238–
565–705	47	48–258			260–
709–885	59	60–324			326–
195–215	16			17–87	89–
219–299	17	21–93			
219–251	17	21–93			95–
255–259	29			87–101	99–
263–283				69–97	99–
287–299	25		25–125		127–
303–375	25	29–137			139–
379–435	29	33–159			
379–431	29	33–159			161–
435				113–161	163–
439–443	29	33–159			161–
447–495	37	41–203			205–
499–587	41	45–225			227–
591–635	49	53–269			271–
639–731	53	57–291			293–
735–915	61	65–335			337–

missible $(v-n,n)$ *and* $v-n\equiv 3\,(\mathrm{mod}\,4)$, $799=15m_0+4\leqslant v-n\leqslant 15m_{11}+4=2059$ *and* $(m_k+4)/(15m_{k-1}+4)\leqslant n/(v-n)\leqslant(11m_k-1)/(30m_k)(\geqslant 0.365)$.

Let $\{m_k\}=59,67,71,79,83,103,107,127,131$. From Lemma 4.26 we have

Lemma 4.32. *If there exists a* $(v,n,4,1)$*-IPMD for admissible* $(v-n,n)$ *and* $m_k\leqslant v-n\leqslant 3m_k$ *and* $n=1,3,\ldots,(m_k-1)/2$, *then there exists a* $(v,n,4,1)$*-IPMD for admissible*

$(v-n,n)$ *and* $v-n \equiv 1\,(\mathrm{mod}\,4)$, $889 = 15m_0+4 \leqslant v-n \leqslant 15m_8+4 = 1969$ *and* $(m_k+1)/(15m_{k-1}+4) \leqslant n/(v-n) \leqslant (11m_k-1)/(30m_k)(\geqslant 0.365)$.

Let $\{m_k\}$ as follows:

$$137, 149, 157, 169, 181, 193, 209, 221, 233, 253, 269, 289, 313, 337, 361, 389, 421,$$

$$457, 497, 541, 589, 641, 697, 757, 821, 893, 973, 1057, 1149, 1249, 1353, 1473,$$

$$1605, 1749, 1905, 2073, 2257, 2461, 2681, 2921, 3185, 3473+4s, \quad s=0,1,\ldots\ .$$

It is easy to see that $\{m_k\}$ satisfy $12m_i+3 \geqslant 11m_{i+1}+8$, that is, $12m_i-5 \geqslant 11m_{i+1}$, and $m_i = 4k+1$, $N(m_i) \geqslant 11$ (see Theorem 2.71 in [7]). From Lemmas 4.19 and 4.20, we have

Lemma 4.33. *There exists a* $(v,n,4,1)$*-IPMD for admissible* $(v-n,n)$ *and* $v-n \equiv 0\,(\mathrm{mod}\,4)$, $1648 = 12m_0+4 \leqslant v-n$ *and* $m_k/(12m_{k-1}+4) \leqslant n/(v-n) \leqslant 5m_k/12m_k$, n *is odd.*

Lemma 4.34. *There exists a* $(v,n,4,1)$*-IPMD for admissible* $(v-n,n)$ *and* $v-n \equiv 3\,(\mathrm{mod}\,4)$, $1647 = 12m_0+3 \leqslant v-n$ *and* $(m_k+2)/(12m_{k-1}+3) \leqslant n/(v-n) \leqslant (5m_k+2)/(12m_k-1)$, n *is odd.*

Let $\{m_k\}$ as follows:

$$127, 131, 139, 151, 163, 167, 179, 191, 199, 211, 227, 243, 263, 283, 299, 323, 343,$$

$$371, 403, 431, 463, 503, 547, 595, 643, 699, 759, 823, 895, 971, 1055, 1147, 1247,$$

$$1355, 1475, 1607, 1751, 1907, 2075, 2259, 2463, 2683, 2923, 3187, 3475+4s,$$

$$s=0,1,\ldots\ .$$

It is easy to see that $\{m_k\}$ satisfy $12m_i+3 \geqslant 11m_{i+1}+8$, that is, $12m_i-5 \geqslant 11m_{i+1}$, and $m_i = 4k+3$, $N(m_i) \geqslant 11$ (see Theorem 2.71 in [7]).

From Lemmas 4.21 and 4.22 we have

Lemma 4.35. *There exists a* $(v,n,4,1)$*-IPMD for admissible* $(v-n,n)$ *and* $v-n \equiv 0\,(\mathrm{mod}\,4)$, $1528 = 12m_0+4 \leqslant v-n$ *and* $(m_k+3)/(12m_{k-1}+4) \leqslant n/(v-n) \leqslant (5m_k+3)/12m_k$, n *is even.*

Lemma 4.36. *There exists a* $(v,n,4,1)$*-IPMD for admissible* $(v-n,n)$ *and* $v-n \equiv 1\,(\mathrm{mod}\,4)$, $1525 = 12m_0+1 \leqslant v-n$ *and* $(m_k+1)/(12m_{k-1}+1) \leqslant n/(v-n) \leqslant (5m_k+1)/(12m_k-3)$, n *is even.*

Combining the above results we have

Theorem 4.37. *The necessary conditions for the existence of* $(v,n,4,1)$-*IPMD are sufficient except for* $(v-n,n)\in\{(4,0),(8,0),(3,1),(7,1)\}$ *and possible excepting* $(v-n,n)\in\{(17,2),(25,2)\}$.

Although we have not constructed $(v,n,4,1)$-IPMDs for $(v-n,n)\in\{(17,2),(25,2)\}$, we believe they exist. Hence, we post the following conjecture.

Conjecture 4.38. The necessary conditions for the existence of $(v,n,4,1)$-IPMD are sufficient except for $(v-n,n)\in\{(4,0),(8,0),(3,1),(7,1)\}$.

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Appendix

In this section, we will directly construct some 4-IPMDs and 4-HPMDs by the easy and effective ways stated in Section 2.

Theorem A.1. *There exist* $(v,n,4,1)$-*IPMDs for the following* $(v-n,n)$.

Proof. Let $X=G=Z_{v-n}$, and $Y=\{\infty_1,\infty_2,\ldots,\infty_n\}$. We present $\mathscr{B}(1)$ as follows. It is readily checked that $\operatorname{dev}\mathscr{B}$ forms the blocks of the IPMD.

$v-n=35,\ n=13,15,17$:
$(-4,1,6,-3;-8,10)^*,(-13,2,17,-6;14,-10)^*,(16,-17),(-9,4),(11,-5),$
$(-12,-14),(12,-2),(13,-1),(-7,-15),(5,9),(7,8),(3,15),(-16,-11).$

$v-n=39,\ n=5,7,9$:
$(10,1,-8,-3;6,-4)^*,(-10,2,14,-6;-18,-17)^*,(-9,8,16,-15),(9,-12,17,-14),$
$(-2,-7,-11,-19),(12,7,5,15),(19,-13,-1,-5),(11,18),(-16,3),(13,4).$

$v-n=39,\ n=13,15,17$:
$(10,1,-8,-3;6,-4)^*,(-17,2,-18,-6;14,-10)^*,(17,9,-12,-14),(4,-5),(11,19),$
$(13,-1),(-15,-2),(12,7),(18,16),(15,-9),(-13,3),(-19,5),(-7,-11),(-16,8).$

$v-n=43,\ n=5$:
$(-4,1,6,-3),(-10,14,2,-6),(17,11,-12,-16),(3,7,15,18),(-1,-5,-13,19),$
$(-15,-9,-2,-17),(9,21,20,-7),(-20,-14,-21,12),(4,8),(10,13),(-8,-18),$
$(-11,16),(5,-19).$

$v-n=43,\ n=7,9,11$:
$(10,1,-8,-3;6,-4)^*,(-10,2,14,-6;-18,-21)^*,(17,-12,-16,11),(3,7,15,18),$

$(12,8),(-1,-5,-13,19),(-15,-2,-17,-9),(21,9,20,-7),(-20,-11),(5,4),$
$(-14,13),(-19,16).$

$v-n=43,\ n=13,15,17$:
$(-4,1,6,-3;-8,10)^*,(-21,2,-18,-6;14,-10)^*,(-16,11,-12,17),(3,7,18,15),$
$(-20,-17),(20,-11),(4,8),(21,9),(16,13),(-2,19),(-5,-15),(-13,5),(-9,-19),$
$(12,-1),(-14,-7).$

$v-n=47,\ n=9,11,13$:
$(-4,1,6,-3;-8,10)^*,(-10,2,14,-6;-18,22)^*,(17,-12,-16,11),(3,19,7,18),$
$(23,15),(-1,-5,20,-14),(-9,-2,-21,-15),(4,-22,5,13),(-20,8),(-11,-23),$
$(12,21),(-7,-13),(16,-17),(-19,9).$

$v-n=47,\ n=15,17,19$:
$(-4,1,6,-3;-8,10)^*,(-10,2,14,-6;-18,22)^*,(17,-12,-16,11),(3,7,19,18),$
$(5,-22),(-1,-23),(15,21),(20,-21),(12,13),(-15,9),(16,-7),(-20,8),(-17,4),$
$(-14,-19),(-2,-13),(-5,23),(-9,-11).$

$v-n=51,\ n=9,11,13$:
$(10,1,-8,-3;6,-4)^*,(22,2,-18,-6;14,-10)^*,(-22,3,-23,-9),(11,-24,-12,25),$
$(16,-19,-2,5),(-7,-16,4,19),(15,7,-21,-1),(12,-11),(-17,9),(23,-14,24,18),$
$(-20,-25),(21,13),(17,8),(-5,-13),(20,-15).$

$v-n=51,\ n=15,17,19$:
$(10,1,-8,-3;6,-4)^*,(-10,2,14,-6;-18,22)^*,(-22,3,-23,-9),(16,-2,5,-19),$
$(11,-24,-12,25),(-25,20),(23,18),(-11,-1),(13,21),(9,24),(-7,-15),(-17,8),$
$(4,19),(-5,-20),(7,17),(-14,-16),(-13,12),(15,-21).$

$v-n=63,\ n=25$:
$(1,3,-1,-3),(5,10,-5,-10),(31,20,-31,-20),(-25,24),(15,-18),(14,-8),$
$(23,-16),(2,6),(21,-12),(25,-15),(-26,-24),(27,-13),(19,28),(-27,-19),$
$(-30,11),(-28,8),(4,17),(29,-7),(-9,-14),(13,26),(9,16),(30,-4),(-6,-21),$
$(-22,-23),(-11,-17),(7,22),(12,18),(-29,-2).$

$v-n=67,\ n=13,15,17,19,21$:
$(2,-1,-2,1;-3,5)^*,(8,-4,-8,4;-12,20)^*,(32,-16,-32,16;19,13)^*,(25,-10),$
$(18,-9,-18,9;-27,-22)^*,(-26,-7,12,21),(33,-5,29,10),(14,6,17,30),(-17,-29)$
$(-24,11,-33,-21),(7,22,15,23),(-11,-6,-19,-31),(31,-25),(27,-20),(26,-15),$
$(-28,-13),(24,-14),(-23,-30),(28,3).$

$v-n=67,\ n=23,25,27,29,31$:
$(1,2,-1,-2;-3,5)^*,(4,8,-4,-8;-12,20)^*,(16,32,-16,-32;19,13)^*,(29,-14),$
$(9,18,-9,-18;-27,-22)^*,(-26,-7,12,21),(-31,25),(31,-24),(-21,10),(14,7),$
$(-11,-15),(-33,23),(-25,11),(-23,-5),(30,-6),(-30,-17),(27,17),(33,-20),$
$(-28,6),(-10,-19),(22,15),(24,26),(-13,-29),(28,3).$

$v-n=33,\ n=4$:
$(-8,10,1,-3),(15,2,-11,-6),(11,7,3,12),(-9,4,-12,-16),(13,-7,-14,8),$
$(6,14,-15,-5),(-4,-10),(9,-2),(5,-1),(16,-13).$

$v-n=33,\ n=6,8,10$:
$(10,1,-8,-3;6,-4)^*,(-11,2,15,-6;14,-10)^*,(11,7,3,12),(-9,4,-16,-12),$
$(-5,-15),(13,-7,8,-14),(5,-2),(9,16),(-1,-13).$

$v-n=33,\ n=12,14,16$:
$(10,1,-8,-3;6,-4)^*,(-11,2,15,-6;14,-10)^*,(-14,-5),(-15,9),(12,13),$
$(11,-1),(16,5),(-16,3),(-7,4),(-2,-13),(-9,8),(-12,7).$

$v-n=37,\ n=6,8,10$:
$(-4,1,6,-3;-8,10)^*,(-10,2,14,-6;-18,-15)^*,(13,4,15,5),(-1,-13,-9,-14),$
$(17,-12,-16,11),(7,18,9,3),(16,-5),(-7,-17),(12,-11),(8,-2).$

$v-n=37,\ n=12,14,16$:
$(10,1,-8,-3;6,-4)^*,(-15,2,-18,-6;14,-10)^*,(13,4,15,5),(-12,9),(11,18),$
$(-17,8),(-16,-7),(16,-1),(7,-2),(17,-5),(-13,-14),(12,-11),(-9,3).$

$v-n=41,\ n=6,8,10$:
$(10,1,-8,-3;6,-4)^*,(-10,2,14,-6;-18,-19)^*,(13,8,15,5),(-1,-11,-13,-16),$
$(17,-12,9,-14),(4,18,-5,-17),(12,-2,11,20),(3,-9),(16,-15),(19,7),(-20,-7).$

$v-n=41,\ n=12,14,16$:
$(-4,1,6,-3;-8,10)^*,(-10,2,14,-6;-18,-19)^*,(15,13,5,8),(-1,-11,-13,-16),$
$(4,-5),(19,7),(16,20),(-17,11),(18,-9),(17,-7),(-20,-2),(9,12),(-14,3),$
$(-12,-15).$

$v-n=53,\ n=8,10,12,14$:
$(10,1,-8,-3;6,-4)^*,(22,2,-18,-6;14,-10)^*,(-13,3,19,-9;-25,-22)^*,$
$(25,-24,11,-12),(9,-1,18,-26),(20,5,-23,-2),(7,16,17,13),(-19,-20,15,24),$
$(-5,-14,-7,26),(23,21),(4,8),(-15,12),(-17,-21),(-11,-16).$

$v-n=53,\ n=16,18,20,22$:
$(10,1,-8,-3;6,-4)^*,(-10,2,14,-6;-18,22)^*,(-13,3,19,-9;-25,-22)^*$
$(25,-24,11,-12),(26,-5,-7,-14),(18,-23),(21,23),(-19,5),(7,8),(-20,-15),$
$(24,9),(-21,-2),(-17,-11),(-1,-26),(20,16),(-16,13),(12,17),(15,4).$

$v-n=57,\ n=8,10,12,14$:
$(-4,1,6,-3;-8,10)^*,(22,2,-18,-6;14,-10)^*,(-17,3,23,-9;28,-22)^*,$
$(25,-24,11,-12),(-1,18,9,-26),(-5,-7,26,-14),(20,5,-23,-2),(7,16,21,13),$
$(-15,24,12,-21),(-27,-20,15,-25),(-28,17),(19,-11),(-19,4),(8,27),$
$(-13,-16).$

$v-n=57,\ n=16,18,20,22$:
$(-4,1,6,-3;-8,10)^*, (22,2,-18,-6;14,-10)^*, (-17,3,23,-9;28,-22)^*,$
$(25,-12,-24,11), (-1,9,-26,18), (-5,-7,26,-14), (24,-13), (16,-15),$
$(-11,-19), (20,5), (19,4), (-28,-20), (-23,13), (12,27), (17,-2), (15,7), (21,8),$
$(-25,-27), (-21,-16)$.

$v-n=69,\ n=10,12,14$:
$(18,-9,-18,9;-27,-24)^*, (32,-16,-32,16;21,11)^*, (5,29,-22,-12),$
$(13,20,-5,-28), (33,-2,-21,-10), (30,-26,-29,25), (6,23,-4,-25),$
$(8,17,-19,-6), (31,-1,4,-34), (2,-13,26,-15), (34,7,-33,-8), (3,19,1,-23),$
$(12,-7), (-20,14), (27,-17), (-14,-31), (10,-11), (15,-3), (22,-30), (24,28)$.

$v-n=69,\ n=24,26,28,30,32$:
$(2,-1,-2,1;-3,5)^*, (8,-4,-8,4;-12,20)^*, (32,-16,-32,16;21,11)^*, (-13,3),$
$(18,-9,-18,9;-27,-24)^*, (6,19,-6,-19), (27,-22), (-29,23), (-33,26),$
$(-28,17), (34,-20), (25,29), (-31,-21), (33,-14), (-30,10), (-34,12), (22,24),$
$(31,-7), (30,13), (-11,-17), (14,15), (-25,-5), (-15,-23), (-10,-26), (28,7)$.

$v-n=73,\ n=6,8,\ldots,12$:
$g^i(-5,1,5,-1;6,-11)^*,\ g^i(10,21,-10,-21),\ g^i(20,-4,-19,3),$
$g^i(32,-12,-31,11),\ g^i(29,34,12,-2),\ g^i(-3,2),$
where $i=0,1,2,\ g=8=5^{24}\ (\mathrm{mod}\ 73)$, 5 is a primitive root of $p=73$.

$v-n=73,\ n=12,14,\ldots,18$:
$g^i(-5,1,5,-1;6,-11)^*,\ g^i(10,21,-10,-21),\ g^i(20,-32),$
$g^i(3,-19),\ g^i(32,-12,-31,11),\ g^i(29,34,12,-2),\ g^i(-3,2),$
where $i=0,1,2,\ g=8=5^{24}\ (\mathrm{mod}\ 73)$, 5 is a primitive root of $p=73$.

$v-n=73,\ n=18,20,\ldots,36$:
$g^i(5,-1,-5,1;-6,11)^*,\ g^i(3,26),\ i=0,1,\ldots,8,$
where $g=2=5^{8}\ (\mathrm{mod}\ 73)$, 5 is a primitive root of $p=73$.

$v-n=77,\ n=18$:
$(36,-26,-1,-9), (25,-5,-18,-2), (-27,-10,-36,-4), (23,-20,5,-8),$
$(-31,37,10,-16), (15,-3,-32,20), (30,-6,13,-37), (-17,-12,3,26),$
$(-34,-24,6,-25), (9,29,12,27), (32,2), (31,-14), (-35,-28), (28,21), (38,18),$
$(14,-13), (-21,19), (34,-30), (-15,7), (-38,-23), (8,24), (-29,16), (-19,-7),$
$(17,35), (1,4), (-11,22), (-22,-33), (33,11)$.

$v-n=109,\ n=18,20,\ldots,36$:
$g^i(2,-1,-2,1;-3,5)^*,\ g^i(6,31,-6,-31),\ g^i(37,-47),\ i=0,1,\ldots,8,$
where $g=16=6^{12}\ (\mathrm{mod}\ 109)$, 6 is a primitive root of $p=109$.

$v-n=109,\ n=36,38,\ldots,,54$:
$g^i(2,-1,-2,1;-3,5)^*,\ g^i(6,36),\ g^i(-6,21),\ g^i(37,-47),\ i=0,1,\ldots,8,$
where $g=16=6^{12}\ (\mathrm{mod}\ 109)$, 6 is a primitive root of $p=109$.

$v-n=113,\ n=14,16,\ldots,28$:
$g^i(2,-1,-2,1;-3,5)^*,\ g^i(6,39,-6,-39),\ g^i(9,52,-9,-52),\ g^i(33,3),\ i=0,1,\ldots,6,$
where $g=49=3^{16}\,(\mathrm{mod}\,113)$, 3 is a primitive root of $p=113$.

$v-n=113,\ n=28,30,\ldots,42$:
$g^i(2,-1,-2,1;-3,5)^*,\ g^i(6,39,-6,-39),\ g^i(9,13),\ g^i(-9,52),\ g^i(33,3),\ i=0,1,\ldots,6,$
where $g=49=3^{16}\,(\mathrm{mod}\,113)$, 3 is a primitive root of $p=113$.

$v-n=113,\ n=42,44,\ldots,56$:
$g^i(2,-1,-2,1;-3,5)^*,\ g^i(6,40),\ g^i(-6,43),\ g^i(9,13),\ g^i(-9,52),\ g^i(33,3),$
$i=0,1,\ldots,6,$
where $g=49=3^{16}\,(\mathrm{mod}\,113)$, 3 is a primitive root of $p=113$. □

Theorem A.2. *There exist $(v,4,1)$-HPMDs for the following types.*

Proof. Let $v-n=4k$, k be odd,
$G_i=\{0+i,4+i,\ldots,4(k-1)+i,\}$, $i=0,1,2,3$, $G_5=\{\infty_1,\infty_2,\ldots,\infty_n\}$, $\mathscr{E}=\{(0,k,2k,3k)+i\colon i=0,1,\ldots,k-1\}$. We take the group
$G=Z_{v-n}$, $H=\{2g\colon g\in G\}$, a subgroup of G, and present $\mathscr{B}(1)$, and $\mathscr{A}(1)$, It is readily checked that $(\mathrm{dev}\,\mathscr{B})\cup(\mathrm{dev}\,\mathscr{A})\cup\mathscr{E}$ forms the blocks of the HPMD.

$v-n=12,3^4n^1,\ n=4$ (Lemma 2.2).

$v-n=12,3^4n^1,\ n=2$:
$(-1^0,-2^1,-3^1,6^0),(-1^1,2^0,5^0,-6^1),(1^0,2^1),(1^1,-2^0);(-5^0,-5^1),(5^1,-3^0);$

$v-n=20,\ 5^4n^1,\ n=1$:
$(-5,10),(2^1,-1^1,-2^0,1^0),(6^1,-3^1,-6^0,3^0),(7^0,-6^1,-7^1,6^0),$
$(-2^1,-9^1,2^0,9^0),(1^1,-3^0,9^1,-7^0),(-1^0,3^1,-9^0,7^1).$

$v-n=20,5^4n^1,\ n=3$:
$(-5,10),(1^1,-3^0),(-7^0,9^1);(-1^0,7^1),(3^1,-9^0),(2^1,-1^1,-2^0,1^0),$
$(6^1,-3^1,-6^0,3^0),(7^0,-6^1,-7^1,6^0),(-2^1,-9^1,2^0,9^0).$

$v-n=20,5^4n^1,\ n=5,7$:
$(1,2,-1,-2)^*,(-5,10),(-9,7),(9,-3),(-6,-7),(3,6).$

$v-n=44,11^4n^1,\ n=6,8,10,\ldots,16$:
$(1,2,-1,-2)^*,(9,18,-9,-18)^*,(6,19,-6,-19)^*,(7,14,-7,-14)^*,(5,10,-5,-10)^*,$
$(22,-11),(3^0,3^1),(-3^1,-3^0);(17^0,17^1),(-17^1,-17^0);$
$(21^0,21^1),(-21^1,-21^0);(13^0,13^1),(-13^1,-13^0);(15^0,15^1),(-15^1,-15^0).$

$v-n=44,11^4n^1,\ n=5,7,9,\ldots,15$:
$(1,2,-1,-2)^*,(9,18,-9,-18)^*,(6,19,-6,-19)^*,(7,14,-7,-14)^*,(5,10,-5,-10)^*,$
$(22,-11),(17^0,17^1,13^0,-3^1),(-17^0,-13^1),(3^1,-13^0);$
$(-15^0,13^1),(-17^1,15^0);(21^0,-15^1),(15^1,-21^0);(-3^0,21^1),(-21^1,3^0).$

$v-n=52, 13^4n^1,\ n=6,8,\dots,18$:
$(1,2,-1,-2)^*,(9,18,-9,-18)^*,(6,23,-6,-23)^*,(7,14,-7,-14)^*,(5,10,-5,-10)^*,$
$(11,12,-11,-12),(26,-13),(-25,19),(-17,-21),(15,3),(-3,-19),$
$(25^0,25^1,17^0,-15^1),(21^0,21^1),(17^1,-15^0).$

$v-n=52, 13^4n^1,\ n=7,9,\dots,19$:
$(1,2,-1,-2)^*,(9,18,-9,-18)^*,(6,23,-6,-23)^*,(7,14,-7,-14)^*,(5,10,-5,-10)^*,$
$(11,22,-11,-22)^*,(26,-13),(3^0,3^1),(-3^1,-3^0);(17^0,17^1),(-17^1,-17^0);(21^0,21^1),$
$(-21^1,-21^0);(19^0,19^1),(-19^1,-19^0);(15^0,15^1),(-15^1,-15^0),(25^0,25^1),$
$(-25^1,-25^0).$

$v-n=68, 17^4n^1,\ n=9,11,\dots,25$:
$g^i(1,2,-1,-2)^*,\ g^j(31,6,-31,-6)^*,\ g=9,\ i=0,1,2,3,\ j=0,1,2,$
$(27,23,3,15)^*,(22,11),(-22,-23),(34,-17),(25,-11),(21,-15),(-3,-27),$
$(-25,-21),(33^0,33^1),(-33^1,-33^0);(29^0,29^1),(-29^1,-29^0).$

$v-n=68, 17^4n^1,\ n=10,12,\dots,24$:
$g^i(1,2,-1,-2)^*,\ g^j(31,6,-31,-6)^*,\ g=9,\ i=0,1,2,3,\ j=0,1,2,$
$(27^0,23^1,3^0,15^1),(11,22),(-22,-23),(34,-17),$
$(-3^1,-15^0),(-11^0,-15^1);(21^0,33^1),(-33^1,-21^0);(33^0,-27^1),(-21^1,15^0);(-33^0,3^1),$
$(-11^1,-27^0);(25^0,21^1),(25^1,-3^0);(-25^0,27^1),(-25^1,23^0);(29^0,29^1),(-29^1,-29^0).$

$v-n=76, 19^4n^1,\ n=10,12,\dots,28$:
$g^i(1,6,-1,-6)^*,(38,-19),\ g^i(7^0,7^1),\ g^i(-7^1,-7^0);\ g=3,\ i=0,1,\dots,8.$

$v-n=76, 19^4n^1,\ n=9,11,\dots,27$:
$g^i(1,6,-1,-6)^*,\ g=3,\ i=0,1,\dots,8,$
$(38,-19),(11^0,11^1,7^0,-29^1);(23^0,35^1),(-33^1,-21^0);$
$(21^0,-7^1),(33^1,29^0);(-29^0,-13^1),(-21^1,-13^0);(37^0,-11^1),(37^1,13^0);$
$(-23^0,21^1),(-37^1,35^0);(-11^0,-35^1),(7^1,-37^0);(-35^0,29^1),(13^1,-7^0);(33^0,-23^1),$
$(23^1,-33^0).$ □

Theorem A.3. *There exist $(v,4,1)$-HPMDs for the following types.*

Proof. We take the group $G=Z_{v-n}$. If $v-n$ is even, we take $H=\{2g\colon g\in G\}$, a subgroup of G. We present $\mathscr{B}(1)$, and $\mathscr{A}(1)$. It is readily checked that $(\mathrm{dev}\,\mathscr{B})\cup(\mathrm{dev}\,\mathscr{A})$ forms the blocks of the HPMD.

$v-n=40, 8^5n^1,\ n=8,10,\dots,16$:
$(1,2,-1,-2)^*,(9,18,-9,-18)^*,(6,13,-6,-13)^*,(3,14,-3,-14)^*,(8,-4),$
$(16,-8),(-12,-16),(12,4),(19,-17),(17,-11),(-19,-7),(7,11).$

$v-n=45, 9^5n^1,\ n=6$:
$(1,3,8,-12),(-1,-21,18,4),(-14,-2,-7,-22),(22,-3,9,17),(11,16,-18,-9),$
$(-8,7,14,-13),(-16,-17),(-19,6),(19,-11),(12,2),(21,-4),(13,-6).$

$v-n=56, 8^7n^1,\ n=6,8,\ldots,18$:
$(1,4,-1,-4)^*, (9,20,-9,-20)^*, (15,24,-15,-24)^*, (8,23,-8,-23)^*,$
$(17,16,-17,-16)^*, (25,12,-25,-12)^*, (2,10,-2,-10), (6,26,-6,-26),$
$(18,22,-18,-22), (3^0,3^1), (-3^1,-3^0); (27^0,27^1), (-27^1,-27^0); (11^0,11^1),$
$(-11^1,-11^0); (19^0,19^1), (-19^1,-19^0); (5^0,5^1), (-5^1,-5^0), (13^0,13^1), (-13^1,-13^0)$.

$v-n=65, 13^5n^1,\ n=24$:
$(7,1), (21,3), (-2,9), (-6,27), (-18,16), (11,-17), (-32,14), (-31,-23), (-28,-4),$
$(-19,-12), (8,29), (24,22), (-7,-9), (-21,-27), (2,-16), (6,17), (18,-14),$
$(-11,23), (32,4), (31,12), (28,-29), (19,-22), (-8,-1), (-24,-3),$
$(13,-26,-13,26)$.

$v-n=8, 2^4n^1,\ n=1,2,3$ (Example 3.10 in [13]).

$v-n=10, 2^5n^1,\ n=0,2$ (Theorem 1.5).

$v-n=10, 2^5n^1,\ n=4$:
$(2,-1), (-2,4), (-3,-4), (3,1)$.

$v-n=12, 2^6n^1,\ n=0,2$ (Theorem 1.5).

$v-n=12, 2^6n^1,\ n=3$:
$(1,-3,-2,4), (-5,-4), (5,3), (2,-1)$.

$v-n=12, 2^6n^1,\ n=4$:
$(5^0,2^1,1^1,4^0), (2^0,1^0), (-4^1,-1^1); (-3^0,4^1), (5^1,-4^0); (3^1,-1^0), (3^0,-5^1)$;

and union the following blocks:

$(\infty_4,2,0,10), (\infty_4,6,4,2), (\infty_4,10,8,6), (\infty_4,3,1,11),$
$(\infty_4,7,5,3), (\infty_4,11,9,7), (\infty_4,0,7,4), (\infty_4,4,11,8),$
$(\infty_4,8,3,0), (\infty_4,1,10,5), (\infty_4,5,2,9), (\infty_4,9,6,1)$.

$v-n=14, 2^7n^1,\ n=0,2$ (Theorem 1.5):

$v-n=14, 2^7n^1,\ n=4$:
$(1,3,-6,2), (5,-4), (6,-1), (-3,-5), (4,-2)$.

$v-n=15, 3^5n^1,\ n=0$ (Theorem 1.5).

$v-n=15, 3^5n^1,\ n=4$:
$(1,2,-6,3), (-7,6), (4,-2), (7,-1), (-3,-4)$.

$v-n=15, 3^5n^1,\ n=6$:
$(-3,2), (6,-4), (-7,4), (3,1), (7,-1), (-6,-2)$.

$v-n=16, 4^4n^1,\ n=0,4$ (Theorem 1.5).

$v-n=16, 4^4n^1,\ n=2$:

$(1^0, 2^1, -1^1, -2^0), (-7^0, -2^1, 7^1, 2^0), (1^1, 6^0, -1^0, -6^1), (-7^1, -6^0, 7^0, 6^1)$
$(-5^0, -5^1), (3^1, 3^0); (5, -3)$.

$v - n = 16, 4^4 n^1,\ n = 6$:
$(-3, 2), (6, -1), (-6, 3), (-2, -7), (7, -5), (5, 1)$.

$v - n = 16, 4^4 n^1,\ n = 3$:
$(1^0, -6^1, 3^1, 2^0), (-1^0, -2^1, -3^1, 6^0), (-7^1, -2^0, 3^0, 6^1)$,
$(2^1, -1^1), (-6^0, -3^0); (-7^0, 5^1), (7^1, 7^0); (-5^0, -5^1), (1^1, 5^0)$.

$v - n = 16, 4^4 n^1,\ n = 5$:
$(1^0, -6^1, 3^1, 2^0), (6^1, -3^1), (-2^0, 7^0); (-7^0, 5^1), (-5^1, 3^0)$;
$(7^1, -1^0), (5^0, 1^1); (2^1, -1^1), (6^0, -5^0); (-2^1, -7^1), (-6^0, -3^0)$.

$v - n = 20, 4^5 n^1,\ n = 0, 4$ (Theorem 1.5).

$v - n = 20, 4^5 n^1,\ n = 2$:
$(-2, -6, -8, -4), (7, 9, -7, -9), (3^0, 6^1, 3^1, 8^0), (-3^0, 2^1, -3^1, 4^0)$,
$(2^0, 1^0), (4^1, -1^1); (1^1, 6^0), (-1^0, 8^1)$.

$v - n = 20, 4^5 n^1,\ n = 4, 6, 8$:
$(1, 3, -1, -3)^*, (2, 9, -2, -9)^*, (8, 6), (-8, -4), (7, -6), (-7, 4)$.

$v - n = 20, 4^5 n^1,\ n = 3, 5$:
$(2, 6, -2, -6)^*, (1^0, 8^1, -1^1, -8^0), (3^0, 4^1, -3^1, -4^0), (9^0, -8^1, -9^1, 8^0)$,
$(9^1, -7^0), (-3^0, 1^1); (7^0, 7^1), (3^1, -9^0); (4^0, -1^0), (-4^1, -7^1)$.

Note that let $B(1) = (2, 6, -2, -6)$, the base block B under $G = Z_{20}$ can be replaced with the following four base blocks under $M = \{4g: g \in G\}$:

$(\infty_4, 0, 2, 8), (\infty_4, 3, 5, -9), (\infty_4, 1, 3, -3), (\infty_4, 2, 4, -2)$;
$(\infty_5, 0, 6, 4), (\infty_5, 3, 9, 7), (\infty_5, 1, -5, -7), (\infty_5, 2, -4, -6)$. □

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